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PTOLEMY'S THEORY OF THE INFERIOR PLANETS

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At its foundation, Ptolemy's astronomy is empirical. Much of the *Almagest* is devoted to the derivation of numerical parameters from observations selected to isolate particular quantities so that they may be found in the most direct way. But in addition the hypotheses, models, of the Sun, Moon and planets were also derived or confirmed by observation although Ptolemy gives only brief descriptions of his procedures without citation of specific observations. In this way, for example, in 9.5 he justifies the use of an epicycle with motion in the positive sense, of increasing longitude, at apogee for the second inequality of the planets, and an eccentric for the first inequality, and in 10.6 he briefly describes his demonstration that the eccentricity must be bisected, separating the centre of uniform motion of the centre of the epicycle from the centre of constant distance. The brevity of the accounts of these demonstrations has been of serious consequence for the history of astronomy because Ptolemy's description of the motions of the bodies in the heavens and his justification of the hypotheses to account for them — both correct, at least in principle, to a high degree of accuracy — were received without further investigation for more than fourteen hundred years, until the time of Tycho and Kepler, during which centuries no significant improvement in accuracy was possible in the absence of such investigations. And for this Ptolemy himself was largely responsible, for he must have believed that the motions had been accurately described and the correct hypotheses discovered once and for all — some were well known and in use before his time in any case — and that, unlike the derivation of parameters where improvements could be made from later observations, no one would ever again have to carry out the labour of deriving or confirming hypotheses. The *Almagest* is to a high degree didactic, and what the reader need never use or do himself is not treated in great detail. Further, the observations used for these demonstrations were of a rough and uncertain kind, such as planetary phases and stations, and in some cases conventional estimates of quantities like lengths and times of retrograde arcs, and the demonstrations, using roundabout methods and approximations, also produced fairly rough numerical results from which mostly qualitative conclusions were drawn, all of which run contrary to Ptolemy's exposition in which proofs and derivations are as definitive and as direct as possible.

However, the demonstrations of hypotheses not only were the foundation of the derivations of parameters in the *Almagest*, but also formed a part of preliminary analyses that provided approximate values of parameters that were then confirmed or improved by the more refined methods set out by Ptolemy at length in his text. And in fact various difficulties in Ptolemy's demonstrations, including the selection and, in some cases, adjustment of observations, depend

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upon such preliminary analyses that showed more or less what to look for both in observations and in the final derivations. It is of course the absence of the preliminary analyses along with the problems of the reported observations and demonstrations that have left Ptolemy open to charges of deception that have been raised at least since the sixteenth century and increasingly so in our own day.[1] However, a consideration of the sorts of analyses that must have preceded the demonstrations in the text, and of the constraints on the observations due to the difficulty of finding the ideal configurations required for the demonstrations, shows that Ptolemy was working within these limitations with extraordinary ingenuity, using less-than-ideal observations of uncertain accuracy to derive or confirm hypotheses and parameters, and conversely using hypotheses and provisional parameters to select and correct these very observations.

If the method appears circular, it is, although 'iterative' would probably be a better word. After all, Ptolemy's only means of verifying the accuracy of observations was consistency, the determination that the result of one demonstration, say, of a parameter, was consistent with other demonstrations of either the same parameter or related parameters, using either the same or different observations and either the same or different procedures. For example, the two sets of eclipses used to derive the radius of the lunar epicycle in 4.6 appear to confirm the same value — although even here some adjustment is likely — and in turn provide the basis for the criticism and correction in 4.11 of times of the eclipses used by Hipparchus for his manifestly inconsistent demonstrations. Such criticism and correction was doubtless applied by Ptolemy, not only to earlier observations, but to his own, and it is probable that many of the reported observations and derivations in the *Almagest* are in their present form refinements of preliminary versions that preceded the demonstrations of the published text. There is important evidence for this conclusion in the recent discovery that the *Canobic inscription*, which contains some parameters differing from the *Almagest*, is actually an earlier work in which Ptolemy set out the principal numerical results of his research before, perhaps several years before, publishing his comprehensive treatise.[2] Whether the demonstrations underlying the *Canobic inscription* were identical to those in the *Almagest* cannot be known, but there is no question that both revision and correction took place in preparing the later work. My guess is that the *Canobic inscription* was based at least in part upon demonstrations preliminary to those in the *Almagest*, and that various difficulties, such as those often noted in the mean motions of the planets, are the result of Ptolemy's neglecting to revise his earlier work to achieve complete numerical consistency, particularly in cases like the mean motions in which complete consistency to six sexagesimal places would be of no practical significance.

Further, and this is very important, the demonstrations or the hypotheses themselves may impose constraints on the parameters that in turn provide a check on the accuracy of observations and an indication of how they are to be corrected after a preliminary derivation. To take a well known example, the procedure for finding the distance of the Sun in 5.15 is so sensitive to minute changes in its parameters — the lunar distance at which the apparent diameters of the Sun and Moon are equal and the apparent diameters of the Moon and shadow at that distance — that the slightest decrease in any of them will cause the solar distance to increase enormously or even become infinite or undefined.[3] Ptolemy surely examined and adjusted the values of the parameters entering into this demonstration, and consequently the observations from which the parameters were derived, in accordance with these severe constraints, for otherwise it is not likely that the demonstration would have reached even a plausible result. An even more remarkable case, as we shall see, is that of Mercury, for which the hypothesis is derived from observations, and then in turn imposes constraints on the parameters that require adjustment of the same observations. The relation of observation and theory in Ptolemy's astronomy is complex, with each used as the basis of, not only the derivation and confirmation, but also the criticism and correction of the other.

The demonstrations for the inferior planets are ideally suited to an investigation of this subject for they appear to be completely empirical in that, not only the parameters, but also the hypotheses themselves are explicitly either derived or confirmed by observations reported in Ptolemy's exposition. After completing the exposition of Mercury, Ptolemy says (10.1), "Such, then, was the method by which we found the hypotheses for the planet Mercury, the size of its anomalies and also the precise amounts of its periodic motions". And likewise, after finding the eccentricities and epicyclic radius of Venus (10.4), "Such, then, is the method by which we determined the type of [Venus's] hypothesis and the ratio of its anomalies".[4] Evidently, he considered the demonstrations to be both derivations of parameters and derivations or confirmations of the hypotheses. And although it does not appear, at least from the text, that there were preliminary analyses of a different sort, we shall see that Ptolemy must earlier have carried out analyses by quite different means, and in this way reached both the necessary structure of the hypotheses and at least preliminary parameters that were then confirmed or refined by the demonstrations he chose to set out.

Greatest Elongations

The demonstrations, both preliminary and final, are determined by the kinds of observations that may be made of the inferior planets, and thus differ greatly from the treatment of the superior planets. In the case of the latter, the most important observations are of oppositions, in which the planet is directly between the Earth and the centre of the epicycle. The equivalent configuration for the inferior planets is inferior conjunction, but since the centre of the epicycle lies in the direction of the mean Sun, and is thus very close to the true Sun, the planet is invisible at inferior conjunction. On the other hand, the planet is clearly visible at greatest elongation from the Sun, whether mean or true, and Ptolemy's demonstrations for Venus and Mercury are based principally upon observations of greatest elongation, or rather of close approaches to greatest elongation.

A greatest elongation is illustrated in Figure 1, omitting the effect of any eccentricity. The centre of the epicycle *C* lies in the direction of the mean sun *S*,

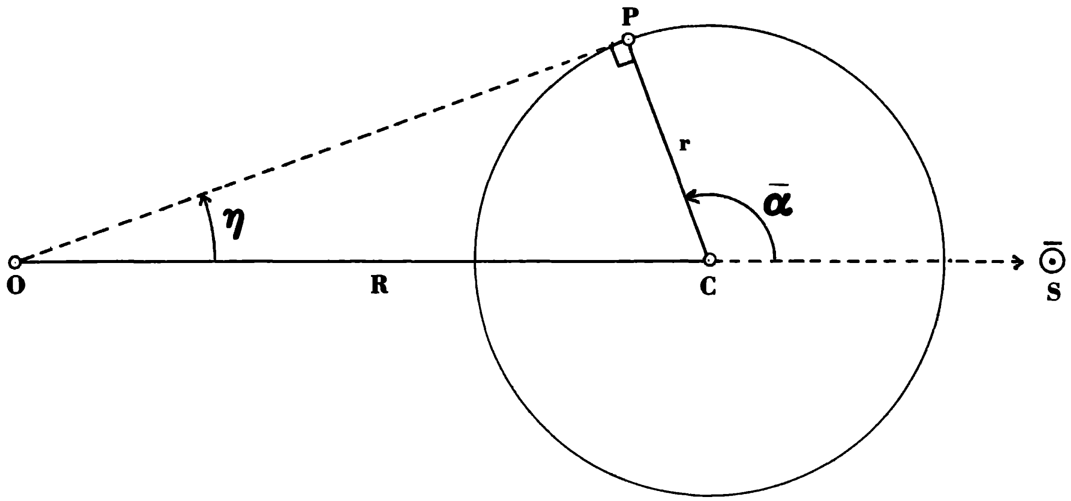

FIG. 1.

and when the planet P is at greatest elongation η, OP is tangent to the epicycle and the angle at P is a right angle. This makes it very simple to estimate the radius of the epicycle $r = R \sin \eta$ and the mean anomaly $\bar{\alpha} = 90° + \eta$. Although greatest elongations were not one of the characteristic phenomena like phases and stations — doubtless because they were not considered ominous — they were observed before Ptolemy. Rough values, probably with respect to the true Sun, are found in a number of ancient sources, and were probably already used before Ptolemy for finding an epicyclic radius since there appears to be no other purpose in noting greatest elongations. For example, Pliny (*Natural history* 2.38–39) gives 46° for Venus and 22° for Mercury, attributing the first to Timaeus and the second to Cedenas and Sosigenes. These are of considerable interest for, letting $R = 60$, we find for the radius of the epicycle,

Venus: $\eta = 46°$, $r = 43;9,37 \approx 43;10$,
Mercury: $\eta = 22°$, $r = 22;28,35 \approx 22;30$,

exactly Ptolemy's parameters. Evidently, he was here confirming earlier estimates, and more importantly, had a provisional radius of the epicycle for his preliminary analyses, which, as we shall see, can be very useful.

The effect of an eccentricity on the magnitude and location of greatest elongation is complex, for it may then take place, not at the tangent point, but some small distance away.[5] The reason is that the change of elongation due to the change of the equation of centre may exceed the change due to the planet's motion on the epicycle. However, for Ptolemy's purposes greatest elongation is understood to take place at the tangent point — compared to the departure from true greatest elongation inherent in his observations the difference is negligible in any case — and his demonstrations depend upon this assumption.

For the demonstrations, Ptolemy uses greatest elongations with the mean Sun in specific locations:

(1) points symmetrical to the apsidal line to find its direction;

(2) the end points of the apsidal line to distinguish apogee and perigee and to find the radius of the epicycle and the eccentricity of the centre of the eccentric;
(3) 90° of mean eccentric anomaly from the apsidal line to find the eccentricity of the centre of the equant circle.

It is also necessary, or at least preferable, that two opposite greatest elongations, morning and evening, take place with the mean Sun in each of these positions. Two important questions are therefore whether during the period of Ptolemy's observations the planets were at greatest elongation when the mean Sun occupied the required positions, or if not, how far they were from greatest elongation, and whether two opposite greatest elongations occurred with the mean Sun in the same position. We can make this examination with the cycles for zodiacal and anomalistic periods used in 9.3 to find the mean motions in longitude and anomaly. The observations of Venus extend from 127 to 140 and of Mercury from 130 to 141, so let us call the outer limits 127–141. For the inferior planets, the cycles are of the form N^y years plus ε^d days contain A^r anomalistic rotations or periods, and each period contains one of each synodic phenomenon, and thus one morning and one evening greatest elongation. Neglecting the small errors ε, the locations of the mean Sun at all elongations of the same kind will divide the zodiac into arcs of $\bar{\delta} \approx 360°/A$, the mean motion of the Sun in each synodic period between successive elongations of the same kind is $\Delta\bar{\lambda}_{\odot} \approx N/A \cdot 360°$ and the motion of the planet on the epicycle in one year is $\Delta\bar{\alpha} \approx A/N \cdot 360°$. The following short cycles are of interest ($\Delta\bar{\lambda}_{\odot}$ and $\Delta\bar{\alpha}$ to the nearest degree):

		A	N	ε	$\bar{\delta}$	$\Delta\bar{\lambda}_{\odot}$	$\Delta\bar{\alpha}$
Venus:		5^r	8^y	-2^d	72°	$1^r + 216°$	225°
Mercury:	*a*	41	13	+3	$8\frac{3}{4}$	114	$3^r + 55$
	b	104	33	−2	$3\frac{1}{2}$	114	$3^r + 55$
	c	145	46	+1	$2\frac{1}{2}$	114	$3^r + 55$

The cycle for Venus is the one used for the mean motions. For Mercury the mean motions were derived from the 46-year cycle *c* that can be divided into two shorter periods *a* and *b*, which combine as $a + b = c$, and it is the 13-year period *a* that is of interest here since it covers about the duration of Ptolemy's observations.

In each cycle there will be A greatest elongations of each kind dividing the zodiac — not successively — into arcs of $\bar{\delta}$ degrees. The required locations of the mean Sun for Ptolemy's demonstrations are separated by arcs of 90°, i.e. apogee, ±90° from apogee, and perigee. Since for Mercury $\bar{\delta} \approx 9°$, at least one greatest elongation of each kind must occur with the mean Sun within $4\frac{1}{2}°$, or $4\frac{1}{2}^d$ of solar motion, of these locations. Conversely, when the mean Sun is at the required locations, the planet must once be within $4\frac{1}{2}^d$, about 14° of motion on the epicycle, of greatest elongation of each kind, which can change the elongation by about 1°. Hence, it will be possible to find the planet quite close to greatest elongation. For Venus, however, conditions are not so favourable.

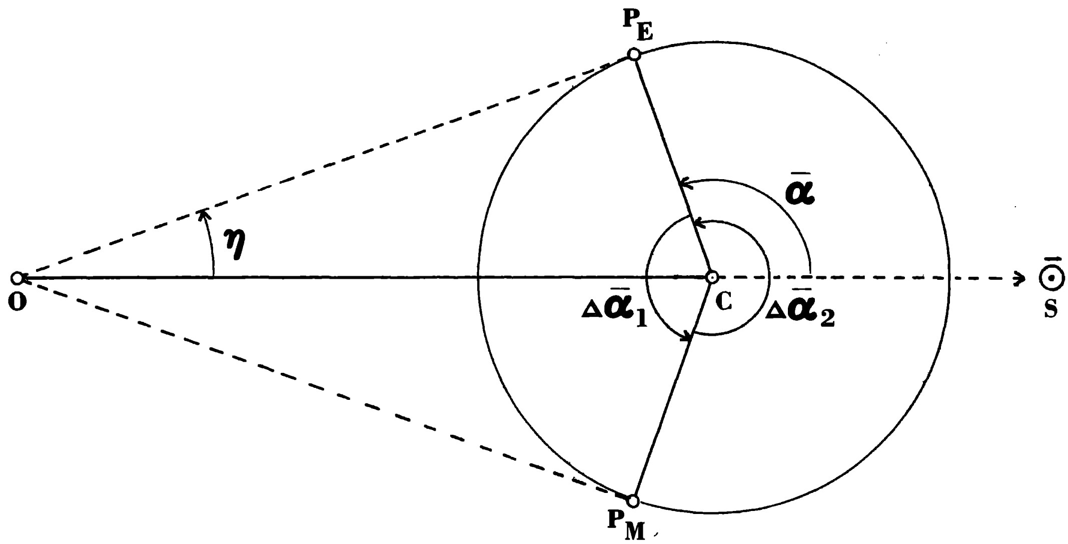

FIG. 2.

There are only five greatest elongations of each kind in each 8-year period, in the next 8-year period the locations of the elongations will shift only about $-2°$, and Ptolemy's observations extend over only 13 years. Since $\bar{\delta} \approx 72°$, at greatest elongation the mean Sun may be as far as 36°, or $36\frac{1}{2}^{d}$ of motion, from a required position, and conversely when the Sun is in such a position the planet may be up to $22\frac{1}{2}°$ of motion on the epicycle from greatest elongation, changing the elongation by $7\frac{1}{2}°$ in the worst case.

The next question is whether two opposite greatest elongations can occur within a cycle with the mean Sun at about the same position. For Mercury, since $\bar{\delta} \approx 9°$ for each kind of elongation, it is obvious that both kinds must occur within arcs of less than $4\frac{1}{2}°$, which is fairly close, but it is not clear that anything like this occurs for Venus where $\bar{\delta} \approx 72°$. Therefore, consider Figure 2 showing arcs of mean anomaly $P_E \rightarrow P_M = \Delta\bar{\alpha}_1$ from evening to morning and $P_M \rightarrow P_E = \Delta\bar{\alpha}_2$ from morning to evening. Since at greatest elongation $\bar{\alpha} = 90° + \eta$, using the conventional values of η, these can be estimated as

Venus:	$\eta = 46°$,	$\bar{\alpha} = 136°$,	$\Delta\bar{\alpha}_1 = 88°$,	$\Delta\bar{\alpha}_2 = 272°$,
Mercury:	22 ,	112 ,	136 ,	222 .

Now, we look for divisions of Venus's 8-year and Mercury's 13-year periods into integral years in which the planet moves through arcs of mean anomaly $\Delta\bar{\alpha}$ — less whole revolutions r — close to these. From $\Delta\bar{\alpha} = n(A/N \cdot 360°)$, where n is an integer $< N$, we find:

Venus:	$n = 2^{y}$	$\Delta\bar{\alpha} = 1^{r} + 90°$	$\approx \Delta\bar{\alpha}_1$	$= 88°$
	6	$3^{r} + 270°$	$\approx \Delta\bar{\alpha}_2$	$= 272$
Mercury:	9	$28^{r} + 138°$	$\approx \Delta\bar{\alpha}_1$	$= 136$
	4	$12^{r} + 222°$	$\approx \Delta\bar{\alpha}_2$	$= 224$

In every case the computed $\Delta\bar{\alpha}$ comes within $\pm 2°$ — about 3^d of anomalistic motion for Venus and 1^d for Mercury — of $\Delta\bar{\alpha}$ between opposite greatest elongations. Consequently, the pattern of elongations and intervals will be:

Venus: $E \rightarrow 2^y \rightarrow M \rightarrow 6^y \rightarrow E \ldots = E_n \rightarrow E_{n+1} = 8^y - 2^d$

Mercury: $E \rightarrow 9^y \rightarrow M \rightarrow 4^y \rightarrow E \ldots = E_n \rightarrow E_{n+1} = 13^y + 3^d$

For Mercury, with 41 elongations of each kind in 13 years, such a recurrence is no wonder, but for Venus with only 5 of each kind in 8 years, it is truly interesting.[6] It means that each evening elongation will be paired with a morning elongation 2 years later in about the same position, which is just what Ptolemy needs, but it also means that all 10 elongations will occur in only 5 narrow regions of the zodiac, with a shift of only $-2°$ in each 8-year cycle, and this makes the problem of finding greatest elongations close to the required positions with respect to the apsidal line even more difficult.

Before leaving the subject of greatest elongations some further points should be made. The intervals, here derived from rough estimates of elongations and short cycles of mean motions, are only mean values. The opposite effects of the equation of centre on the true morning or evening elongation can change the interval $E \rightarrow M$ or $M \rightarrow E$ by as much as 6^d for Mercury and fully 16^d for Venus. Further, it is as difficult to determine by observation the exact time of a greatest elongation as of a station. Near greatest elongation the planet moves with about the same speed as the Sun, about 1° per day, but the elongation changes slowly, by 0;5° in $\pm 2^d$ for Mercury and $\pm 6^d$ for Venus, and in no way could Ptolemy estimate the time closer than these limits. Hence the selection of a particular date for true greatest elongation would be arbitrary in any case. Ptolemy appears to have had observational records of all elongations of Venus and a considerable number of Mercury for the period 127–41. These records contained, not just the date of greatest elongation, which Ptolemy surely knew could not be estimated so closely, but observations of the position of the planet for several days, for Venus as many as 30 days, around the time of greatest elongation. Since it is highly unlikely that the planet is at greatest elongation and the mean Sun at the required position on the same day, Ptolemy takes a date on which the mean Sun is as close to the required position, for this is the more important condition, and the planet as close to greatest elongation, as can be found during the period of his observations. In every case his dates of elongations meet these criteria—during the period 127–41 there are no better dates for Venus and a single slightly better date for Mercury in June of 141 which may be too late—and this shows that his records must have been extensive enough for him to select exactly what he needed. Unfortunately, Ptolemy explains none of this, and even though he must have known that the planet was not at true greatest elongation at the reported time of the observation, he calls these configurations "greatest elongation", probably because the planet was very close and because the observation is applied as though it were, but also because, as we shall see, the observation has been adjusted to the position the planet would have if it were at greatest elongation, with greatest elongation interpreted as the tangent point on the epicycle. This has led to much confusion and the belief that Ptolemy made large errors in finding the time of greatest elongation. The observations do have some notable errors, which we shall discuss, but the departure from the date of true greatest elongation is not an error, but a compromise necessitated by the positions of the mean Sun required for the demonstrations.

In the following exposition, we shall both review Ptolemy's demonstrations in the *Almagest* and also consider the preliminary analyses that preceded them which will, we believe, make his own demonstrations clearer and explain a number of their difficulties.[7]

Venus

The observations of the inferior planets are among the most interesting in the *Almagest* because so much detail is given. Ptolemy seems to paraphrase or even quote the actual report, as he wrote it down or received it, and he then reduces it to find the elongation from the mean Sun. Eight observations are used to derive the hypothesis and its parameters, and two, not of greatest elongations, to correct the mean anomaly. Three (1, 3, 5) were given to Ptolemy by "Theon the mathematician", apparently an associate in Alexandria about whom nothing is known. These, and two by Ptolemy (2, 4), are of configurations of Venus with fixed stars without use of the armillary—one wonders if Ptolemy did not even prefer this when possible—and four (6–9) were made by Ptolemy with the armillary set on a fixed star. An ancient report (10) by Timocharis appears to be an observation of an occultation of a star by Venus, although no such occultation occurred. We shall not consider the observations (9–10) used to correct the mean anomaly.

As an example of a report and reduction of an observation of a configuration, here is one (3) by Theon that is particularly interesting because it includes an (illusory) apparent diameter of Venus. The observation in 10.1 is from Hadrian 12 Athyr 21/22 (127 Oct 11/12):

> Venus as a morning-star had its greatest elongation from the Sun when it was to the rear [east] of the star on the tip of the southern wing of Virgo [β Vir] by the length of the Pleiades [$1\frac{1}{2}°$], or less than that amount by its own diameter [$\frac{1}{12}°$!]; and it seemed to be passing the star one Moon [$\frac{1}{2}°$] to the north.

For the given date at 5–6 a.m. Venus and β Virginis are well above the eastern horizon and the Pleiades are visible above the western horizon for Theon's comparison, which does not seem easy to make. The longitude of the star in Ptolemy's catalogue is ♌ 29°, and for the 10 years preceding Antoninus 1, the epoch of the star catalogue, Ptolemy subtracts $\frac{1}{12}°$ to give ♌ $28\frac{11}{12}°$. Adding to this the length of the Pleiades $1\frac{1}{2}°$ for the distance between the star and Venus gives ♍ $0\frac{5}{12}°$, but Ptolemy gives ♍ $0\frac{1}{3}°$, meaning that he has subtracted $\frac{1}{12}°$ or 5′ for the apparent diameter of Venus, a very high estimate.[8] By computation the mean Sun was at ♎ $17\frac{26}{30}°$, and thus the elongation of Venus as a morning star was $47\frac{16}{30}°$. Note the fractions rather than sexagesimals; these are characteristic of Ptolemy's reports and reductions of observations, and probably reproduce the

TABLE 1.

No.	Date	$\bar{\lambda}_{\odot}$	Obs η	Com η	Mod η	Date	η_{max}
1.	132 Mar 8 E	344;15°	47;15°	47;10°	46;50°	Feb 21	47;58°
2.	140 Jul 30 M	125;45	−47;15	−46;32	−46;11	Jul 14	−46;55
3.	127 Oct 12 M	197;52	−47;32	−47;24	−47; 7	Sep 22	−48;21
4.	136 Dec 25 E	272; 4	47;32	47;47	46;52	Dec 13	47;31
5.	129 May 20 M	55;24	−44;48	−44;10	−44;36	May 6	−44;52
6.	136 Nov 18 E	235;30	47;20	45;54	45;34	Dec 13	47;31
7.	134 Feb 18 M	325;30	−43;35	−43;32	−44;33	Feb 15	−44;33
8.	140 Feb 18 E	325;30	48;20	48;16	47;59	Feb 19	47;59

form of his own records. By recomputation from Ptolemy's theory for 127 Oct 12 6 a.m., $\bar{\lambda}_{\odot}$ = ♎ 17;51°, λ = ♍ 0;27°, and η = −47;24°, which differs by +0;8° from the reduction (or +0;3° if the diameter of Venus were not deducted).

Observations 1–8 are summarized in Table 1. For each we give the date, time as morning M or evening E, the mean longitude of the Sun $\bar{\lambda}_{\odot}$, and the elongation of Venus η, all as reported by Ptolemy although substituting sexagesimal for common fractions. These are followed by recomputation from Ptolemy's theory and from modern theory of the elongation η for the date of the

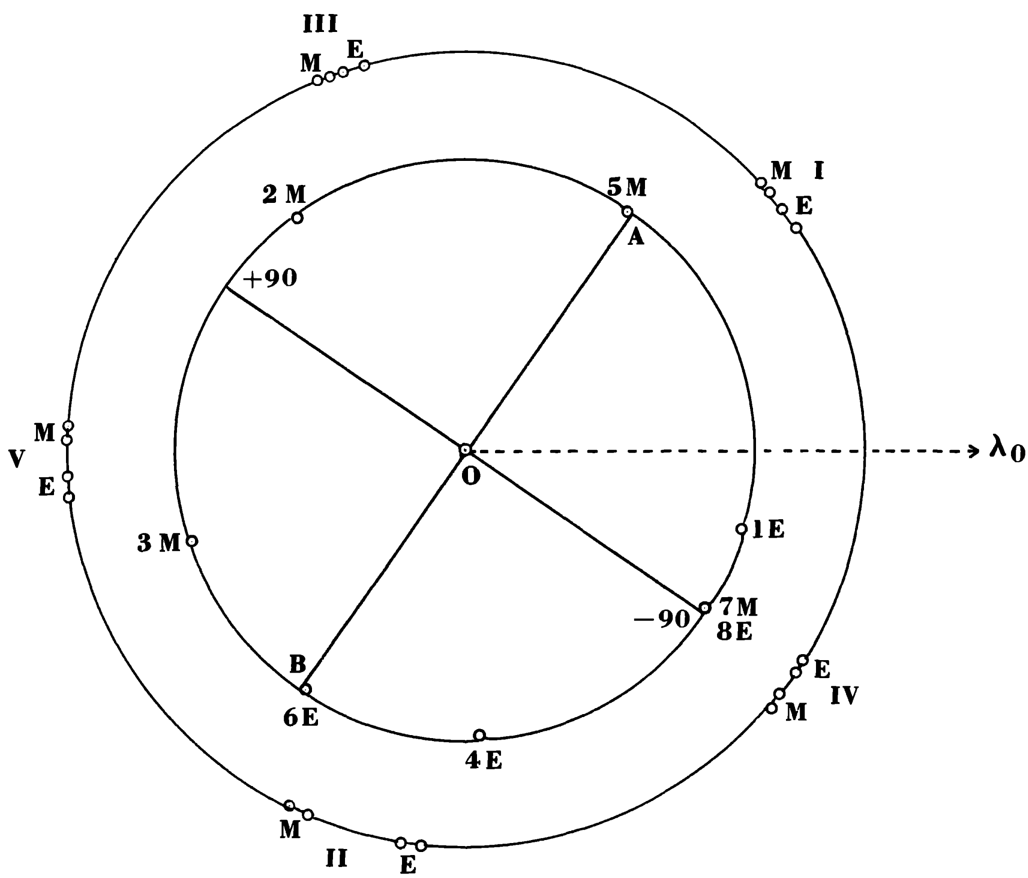

FIG. 3.

TABLE 2.

Region	E: $\bar{\lambda}_{\odot}$	η_E	M: $\bar{\lambda}_{\odot}$	η_M	$\eta_E - \eta_M$	Ptolemy
I	37–34°	$46\frac{1}{6}$°	42–40°	$-44\frac{5}{6}$°	91°	$89\frac{5}{6}$°
II	263–60	$47\frac{1}{2}$	246–43	−47	$94\frac{1}{2}$	$94\frac{5}{6}$
III	108–05	$44\frac{2}{3}$	112–10	−47	$91\frac{2}{3}$	$90\frac{1}{2}$
IV	328–26	48	322–20	$-44\frac{1}{2}$	$92\frac{1}{2}$	$91\frac{2}{3}$
V	186–83	$45\frac{1}{6}$	178–76	$-48\frac{1}{3}$	$93\frac{2}{3}$	$93\frac{2}{3}$

observation, taking the time as 6 a.m. for M and 6 p.m. for E, and then the date of true greatest elongation and its value η_{max}.[9] Note that elongations 4 and 6, which are 37^d apart, are respectively 12^d after and 25^d before the same true greatest elongation — proof, if any were needed, that Ptolemy knew very well that these were not true greatest elongations — and except for 7 and 8, which are very close to opposite greatest elongations six years apart, the other intervals from true greatest elongations are 14^d–20^d. The locations of the mean Sun in these elongations are shown on the inner circle of Figure 3 marked 1–8 and E for evening and M for morning elongation, and the outer circle shows the locations of the mean Sun in all twenty true greatest elongations — computed by modern theory and reduced by 1° to Ptolemy's equinox — in the two complete 8-year cycles of the period 127–41 inclusive. The locations fall into five regions numbered I–V in the order of the successive elongations of the same kind about 216° and 584^d apart, and each region has two E and two M elongations. The figure also shows Ptolemy's apsidal line *AB* and the points ±90° from apogee required for his demonstrations, the distances of which, and the distances of the mean Sun in Ptolemy's observations, from the true greatest elongations can easily be seen.

Ptolemy demonstrates the direction of the apsidal line using pairs of equal and opposite elongations 1–2 and 3–4. However, before selecting such observations from his records, particularly because they were not true greatest elongations, he already had a provisional direction of the apsidal line with a distinction of apogee and perigee. Although he does not set out the observations, he remarks (10.2) that nowhere does he find the sums of opposite elongations to be less than in Taurus or greater than in Scorpio — indeed, the sums of opposite elongations is the best criteria for locating apogee and perigee — and he could determine this with the true greatest elongations from, say, 127–41 shown in the figure or even with the elongations of one 8-year period. In Table 2 we give for each true greatest evening and morning elongation in each region, the mean longitude of the Sun $\bar{\lambda}_{\odot}$, the elongation η_E or η_M, and the sum of opposite elongations $\eta_E - \eta_M$ to the nearest $\frac{1}{6}$°. The last column shows $\eta_E - \eta_M$ computed from Ptolemy's theory of Venus — $\bar{\lambda}_{\odot}$ also differs slightly — for the same period 127–41. Clearly I in Taurus is near apogee and II, although in Sagittarius rather than Scorpio, is near perigee, and since I and II are more than 180° apart, the apogee must be at a longitude > I, or the perigee < II, or both are true (as in fact it turns out).

Having made the important provisional distinction of apogee and perigee, Ptolemy can look through his records of observations near greatest elongation

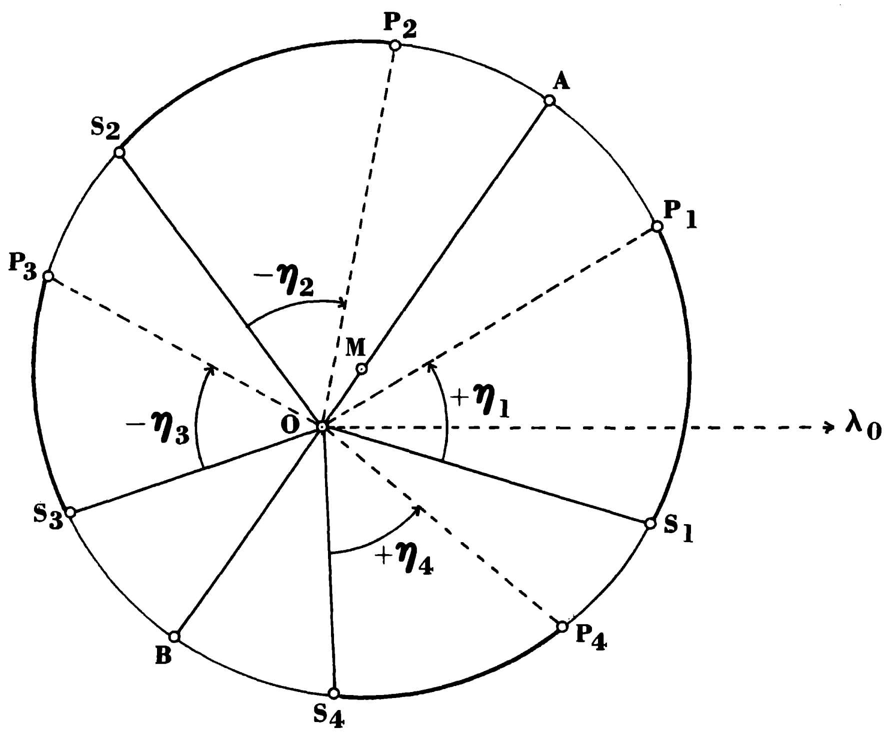

FIG. 4.

to find pairs of equal and opposite elongations, for then the mean Sun is symmetrical to the apsidal line, which may be found by bisecting the arc of longitude between the positions of the mean Sun. This is the inverse application of a demonstration in 9.6 that at equal mean longitudinal distances on either side of apogee or perigee, the equations of centre and the maximum equations of the anomaly are equal. There are in fact two positions on either side of the apsidal line where the greatest elongations, which are the sums of the equation of centre and the maximum equation of the anomaly, are equal — they are not easy to find for their exact locations depend upon both the equation of centre and the distance of the epicycle — and the wrong pairing would not be symmetrical to the apsidal line.[10] However, the restriction of greatest elongations of Venus to just five regions over a period of a number of years, the distribution of the four selected elongations, and Ptolemy's provisional location of apogee and perigee preclude this ambiguity. Ideally, the demonstration should be done from equal sums of opposite elongations at positions on either side of the apsidal line, which are uniquely symmetrical, but these too are precluded by the limited locations of greatest elongations. Hence Ptolemy finds elongations 1–2 and 3–4 — in each pair the elongation is equal and opposite — shown in Figure 4 in which the mean Sun is *S*, the direction of the planet *P*, and the elongation η. Since $\eta_1 = -\eta_2$ and $-\eta_3 = \eta_4$, each pair S_1S_2 and S_3S_4 is symmetrical to the apsidal line, and thus, bisecting the arcs between them, the apsidal line passes through ♉ 25° and 25°. Now, from the sums of true greatest elongations, Ptolemy already knows that the apogee is in Taurus and the perigee in Scorpio, but to confirm the distinction, he takes elongations 5–6

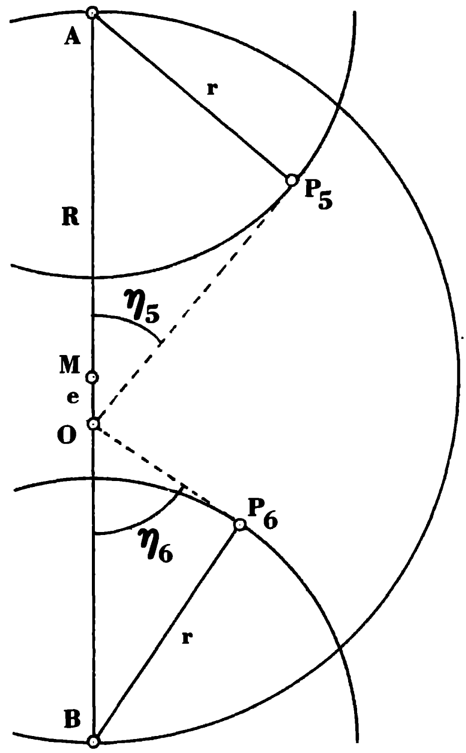

FIG. 5.

in which $\bar{\lambda}_{\odot 5} \approx$ ♉ 25° and $\bar{\lambda}_{\odot 6} \approx$ ♏ 25°. Since $\eta_5 < \eta_6$, the apogee must be at ♉ 25° and the perigee at ♏ 25°.

Elongations 5 and 6 are also used to derive the radius of the epicycle and the eccentricity determining distance. Here and in the following demonstrations, we use the absolute values of the elongations. In Figure 5, since r is constant

$$r = (R + e) \sin \eta_5 = (R - e) \sin \eta_6,$$

from which it follows that

$$\frac{e}{R} = \frac{\sin \eta_6 - \sin \eta_5}{\sin \eta_6 + \sin \eta_5}.$$

Ptolemy, solving in separate steps from $\eta_5 = 44;48°$ and $\eta_6 = 47;20°$, finds that where $R = 60$, $e \approx 1\frac{1}{4}$ and $r = 43\frac{1}{6}$. Note that e is one-half the solar eccentricity of $2\frac{1}{2}$ found in 3.4 and r exactly the radius following from the pre-Ptolemaic estimate $\eta = 46°$. Neither is a coincidence.

To find the eccentricity determining direction, Ptolemy uses opposite elongations 7 and 8, both with the mean Sun at about $-90°$ from apogee where the equation of centre is near maximum. These are also the closest of all Ptolemy's observations to true greatest elongation. The configuration is shown in Figure 6 in which the angle at E is 90° and the elongations η_7 and η_8 are separated by the parallel direction OS from the Earth to the mean Sun. The equation of the anomaly c_a and equation of centre c_c are

$$c_a = \tfrac{1}{2}(\eta_8 + \eta_7), \quad c_c = \tfrac{1}{2}(\eta_8 - \eta_7),$$

from which

$$OC = r/\sin c_a, \quad OE = OC \sin c_c.$$

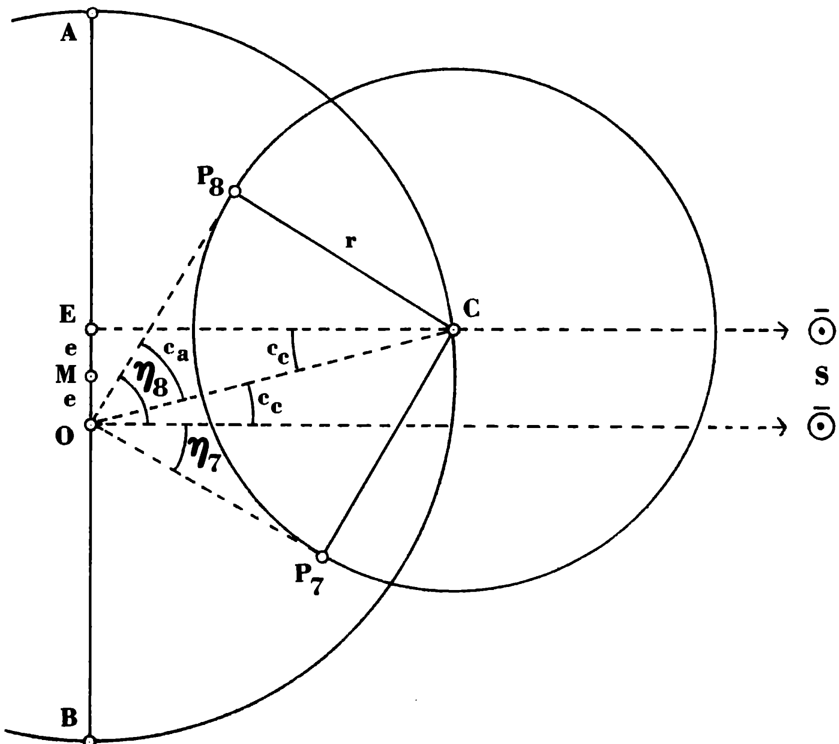

FIG. 6.

From $\eta_7 = 43;35°$ and $\eta_8 = 48;20°$, Ptolemy finds

$$c_a = 45;57,30°, \quad c_c = 2;22,30°, \quad OC = 60;3, \quad OE \approx 2\tfrac{1}{2}.$$

Here we note that $OE = 2e$ is equal to the solar eccentricity and is twice the eccentricity e determining distance.

The hypothesis of Venus that has been established is shown in Figure 7. The observer is at O, the centre of the eccentric M, the equant centre E, and A the apogee of the eccentric on which the centre C of the epicycle moves through the mean eccentric anomaly $\bar{\kappa}$ uniformly about E such that the direction EC is parallel to the direction from the Earth to the mean Sun S. The planet P moves on the epicycle of radius r through the mean anomaly $\bar{\alpha}$ uniformly with respect to the mean apogee F lying in the direction ECS. Where the radius of the eccentric $R = 60$, the simple eccentricity of the centre of the eccentric $OM = e = 1;15$, the double eccentricity of the equant centre $OE = 2e = 2;30$, and the radius of the epicycle $r = 43;10$. The equation of centre c_c is subtended by the double eccentricity and the equation of the anomaly c_a is subtended by the radius of the epicycle.

Although apparently straightforward, the derivation of Venus's parameters, which is also supposed to be a confirmation of its hypothesis, raises serious questions. Put simply, Ptolemy could not have done it as he explains, and at least one of the observations must have been altered. The selection of observations is determined by the positions of the mean Sun, and the consequent compromises with the time of true greatest elongation are more or less harmful. For elongations 7 and 8 at $-90°$ from the apsidal line, used to find $2e$, the positions are favourable and the compromises small, but for elongations 5 and 6

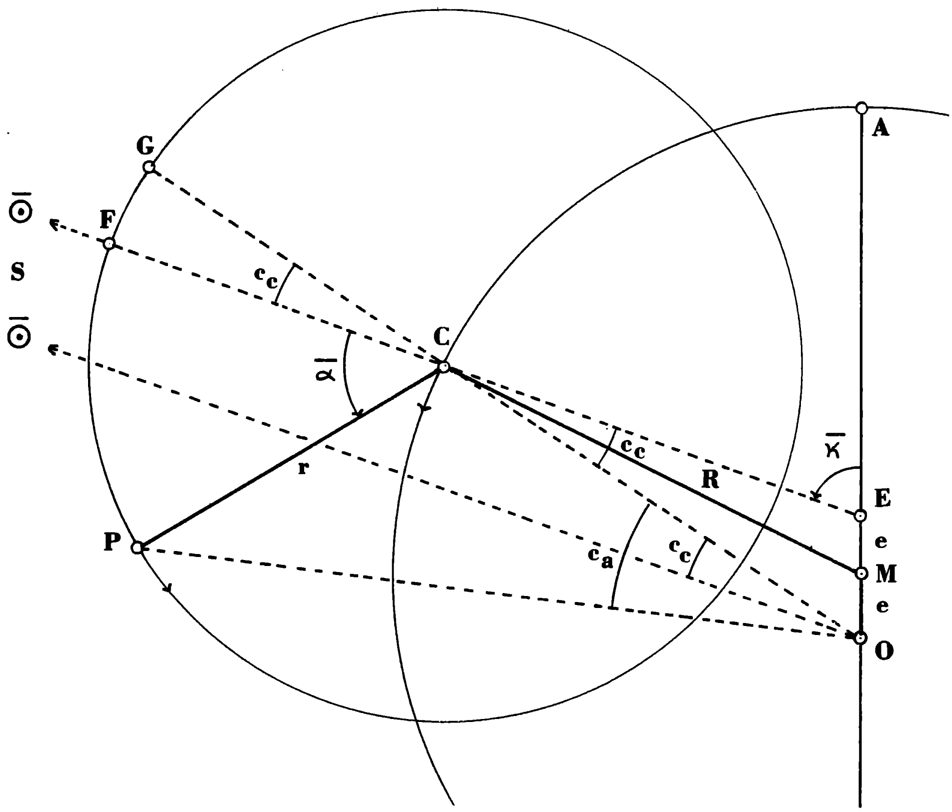

FIG. 7.

in the apsidal line, 6 in particular is much too far away from true greatest elongation to give anything close to the elongation reported by Ptolemy. Ptolemy claims 47;20°, computation gives 45;34°, a difference of nearly 2° which is intolerable — even Ptolemy's theory gives 45;54° — and the true greatest elongation 25 days later of 47;31° shows far too small a difference from 47;20° for so long an interval. My guess is that Ptolemy knew the elongations he required in the apsidal line—they are too critical to compromise—and arranged elongation 6 accordingly.

In fact, I believe that his original analysis was of this sort:

(1) From the minimum and maximum sums of opposite greatest elongations in regions I and II, the apogee is approximately located near the end of Taurus and the perigee near the end of Scorpio or the beginning of Sagittarius.

(2) From the traditional greatest elongation $\eta = 46°$, the radius of the epicycle is $r = R \sin 46° \approx 43;10$ subject to confirmation in (5).

(3) From elongations 7 and 8 in Aquarius, very close to true greatest elongation and about 90° from the apogee in Taurus, the double eccentricity of the equant is $2e = 2;30$, exactly that of the Sun. It may also be pertinent that in Indian astronomy the eccentricity of Venus and the Sun are usually equal, a tradition that may have Greek antecedents (although none is known).

(4) The precise location of the apsidal line is taken as 90° from elongations 7 and 8, and rounded to ♉ 25° and ♏ 25°, for it does seem remarkable luck to

have true greatest elongations so close to 90° from the apsidal line. Hence it is observations 7 and 8 that determine the exact location of the apsidal line, something that could not be determined from the distribution of the true greatest elongations.

(5) The eccentricity $e = 1;15$ determining distance may have been taken as half $2e = 2;30$ by assumption that the bisection applies also to Venus. Or there is another way that confirms the bisection independently. Use the true greatest elongations from regions I and II near the apsidal line as though they were in the apsidal line. For example, from the modern computation of η_{max} let $\eta_5 = 44;50°$ and $\eta_6 = 47;30°$, and from Ptolemy's procedure it follows that $e = 1;20$ and $r = 43;15$, the former close enough to a bisection of $2e$ to confirm $e = 1;15$ and the latter close enough to the traditional 43;10 to confirm it. The elongations in the apsidal line are then

$$\eta_5 = \sin^{-1}\frac{r}{R+e} \approx 44;48°, \quad \eta_6 = \sin^{-1}\frac{r}{R-e} \approx 47;20°.$$

The observation of elongation 6, and perhaps also elongation 5, must have been adjusted to fit these results which are correct in the sense that, given that e and r have been independently confirmed, *if* there were true greatest elongations at apogee and perigee, these would be their values. Essentially, this is a correction for the fact that the planet was not at greatest elongation at the time of the observation. And it is for this reason that Ptolemy calls the observations, evidently now with some justice, "greatest elongations". We shall discuss such adjustments further in our consideration of Mercury where the reasons for them are even more compelling.

Mercury

In the case of Mercury, Ptolemy's empirical derivation of hypothesis and parameters yields its most remarkable and enigmatic results, for some of the observations raise serious problems and the hypothesis derived from them turns out to differ from that of all the other planets. Apparently it was not always so. In the *Canobic inscription* Mercury is assigned an epicyclic radius $r = 22;30$ and an eccentricity $e = 2;30$ without indication that there is anything unusual about its hypothesis.[11] The radius of the epicycle, which is confirmed in the *Almagest*, follows from the greatest elongation of 22° that, like the 46° elongation of Venus, antedates Ptolemy. The eccentricity is equal to the eccentricity of the Sun and the double eccentricity of Venus, and thus, assuming a hypothesis like the other planets, the double eccentricity and equation of centre will be twice that of the Sun and Venus. However, Ptolemy later changed his mind, as he says himself in 4.9 after explaining improvements in his method of finding the mean argument of lunar latitude:

> We have done something similar with the hypotheses for Saturn and Mercury, changing some of our earlier, somewhat incorrect assumptions because we later got more accurate observations. For those who approach this science in a true spirit of enquiry and love of truth ought to use any

new methods they discover, which give more accurate results, to correct not merely the ancient theories, but their own too, if they need it. They should not think it disgraceful, when the goal they profess to pursue is so great and divine, even if their theories are corrected and made more accurate by others beside themselves.

In the *Almagest* there are two changes in the eccentricity: (1) the centre determining direction, the equant centre, is half rather than twice the eccentricity determining distance, and (2) the centre determining distance, the centre of the eccentric, is made to rotate in a small circle such that the least distance of the epicycle is not reached opposite the apogee, but at two points 120° from the apogee. In his exposition Ptolemy derives each element of this peculiar hypothesis observationally, evidently from the "more accurate observations", although just as for Venus some of his demonstrations seem to depend upon prior analyses that differ from those he chose to set out. Another distinctive feature of his treatment of Mercury is that it is for Mercury alone that he demonstrates from observations about 400 years before his time that its apsidal line moves 1° in 100 years with respect to the equinoxes, just as the fixed stars, and thus is sidereally fixed. This conclusion is then applied to the other planets without explicit proof, for Ptolemy simply remarks (9.7) that "the phenomena associated with the other planets individually fit [this assumption]".

The observations of Mercury are on the whole closer than those of Venus to greatest elongation when the mean Sun is in the positions required for the derivation of parameters, the reason being, as we have noted, that in 13 years Mercury has 41 greatest elongations of each kind dividing the zodiac into arcs of only about 9°, rather than 72° for Venus, and thus when the Sun is in the required positions the planet is not more than about $4\frac{1}{2}$ days from greatest elongation. And as we have also noted, the observations selected by Ptolemy are the optimal configurations that occur for his demonstrations during the period 127–41. In all, Ptolemy uses 16 observations of Mercury, more than of any other planet. Eight are used to derive the hypothesis and its elements, six ancient observations to find the direction of the apsidal line 400 years earlier, and two to correct the mean anomaly. Of the first eight, seven were made by Ptolemy with the armillary and one (7) by Theon as a distance from Regulus for which no instrument is specified (although interestingly enough it is the most accurate of all the observations). Ptolemy explains (9.8) that the armillary is particularly valuable for observing Mercury for, although nearby stars are seldom visible where Mercury can be observed, not far from the horizon near dawn or dusk, the armillary may be set on more distant bright stars — Ptolemy uses Aldebaran, Regulus, and Antares — that are visible.

Table 3 arranged like that for Venus, gives for observations 1–8 the date, whether morning M or evening E, the mean longitude of the Sun $\bar{\lambda}_{\odot}$ and the elongation η as reported by Ptolemy and as computed from Ptolemy's theory, followed by recomputation from modern theory of the elongation η, and then the date and value of true greatest elongation η_{max}. Recall that Mercury's 13-year period is divided into two intervals between opposite greatest elongations with the Sun at about the same longitude — 9 years from E to M and 4 years from M to E. It is this division that makes possible Ptolemy's observations of opposite pairs in the required positions, 1–4 and 7–8 each 9 years apart and 2–3 4 years apart. The intervals from true greatest elongation are all 4 days or less, as would be expected, except for 2 which is 6 days.

Just as for Venus, the direction of the apsidal line is established from two pairs of equal opposite elongations, $\eta_1 = -\eta_2 = 21;15°$ and $\eta_3 = -\eta_4 = 26;30°$, so that bisecting the arcs between the mean longitudes of the Sun, the apsidal line passes approximately through ♈ 10° and ♎ 10°. Here again one wonders if a provisional direction was not first reached from an analysis of a larger number of equal elongations or better, sums of opposite greatest elongations in the same locations. We have attempted this by recomputing from modern theory all greatest elongations in the years 127–41 — there are 96 of them, 48 of each kind — and bisecting the arcs between the mean longitude of the Sun where the sums of opposite elongations are nearly equal. The result is that the apsidal line passes through ♉ 14° and ♏ 14° $\pm$ 4°, while accurately it should be ♉ 18° and ♏ 18°, which is close enough to show that the method works in principle. (The apsidal line with respect to the mean Sun passes through ♉ 20° and ♏ 20°, but the difference hardly matters.) It is not immediately clear how Ptolemy found a direction nearly 40° in error. Although it is suggestive that elongations 1–4 are all too large by about 1°–1½°, and in fact the pairs 1–2 and 3–4, although close, are not equal, it will soon be evident that these errors are not by themselves the cause of the mislocation of the apsidal line, which in turn has serious consequences for the entire theory of Mercury. Ptolemy next shows from six ancient observations that 400 years earlier the apsidal line passed through ♈ 6° and ♎ 6°, a difference of 4°, or 1° in 100 years, exactly the rate of the precession. Since Ptolemy probably did not analyse these observations until after he developed his theory of Mercury, we shall consider them later.

Again as for Venus, observations 5–6, very nearly at either end of the apsidal line, distinguish the apogee from the point opposite the apogee. Since $\eta_5 = 19;3°$ is less than $\eta_6 = 23;15°$, the apogee is at ♎ 10° and the opposite point at ♈ 10°. However, ♈ 10° is not the perigee, that is, the closest approach of the centre of the epicycle, for the sum of opposite elongations is a maximum $\pm 120°$ from the apogee rather than 180° where it would in principle be $2\eta_6$ since in the apsidal line the opposite elongations are equal. The evidence is from the first four

TABLE 3.

No.	Date	$\bar{\lambda}_{\odot}$	Obs η	Com η	Mod η	Date	η_{max}
1.	132 Feb 2 E	309;45°	21;15°	20;56°	19;46°	Feb 1	19;50°
2.	134 Jun 4 M	70; 0	−21;15	−21;18	−19;52	Jun 10	−21;20
3.	138 Jun 4 E	70;30	26;30	25;52	25;35	Jun 8	25;53
4.	141 Feb 2 M	310; 0	−26;30	−26;35	−25;15	Feb 1	−25;17
5.	134 Oct 3 M	189;15	−19; 3	−18;33	−19;35	Sep 29	−20;13
6.	135 Apr 5 E	11; 5	23;15	22;58	22;11	Apr 3	22;19
7.	130 Jul 4 E	100; 5	26;15	26;29	26;14	Jul 3	26;15
8.	139 Jul 5 M	100;20	−20;15	−20;25	−19;47	Jul 8	−20;19

elongations, now considered as two pairs of opposite elongations, 1–4 near ♒ 10° and 2–3 near ♊ 10°, for, taking the sums of the absolute values in each pair,

$$\eta_1 + \eta_4 = \eta_2 + \eta_3 = 21;15° + 26;30° = 47;45° > 2\eta_6 = 46;30°.$$

This truly remarkable result is interpreted as showing that the centre of the epicycle is closer to the Earth at ±120° from the apogee than at 180°, which means that the path of the centre of Mercury's epicycle about the Earth cannot be a circle, nor indeed can it be a circle about any fixed point. As we shall see, Ptolemy's hypothesis is designed accordingly. This is empiricism with a vengence.

What went wrong? One might guess that the problem lies with the errors in elongations 1–4, which are all too large. But in fact there is the far greater difficulty that the morning elongation at ♈ 10° opposite to evening elongation 6 could not be observed because at the latitude of Alexandria Mercury has no morning rising in Aries, that is, it is invisible all the way from evening setting to its next evening rising. Likewise, in Libra the evening rising is nearly invisible, so Mercury can be seen only as a morning star, and this is why Ptolemy gives only an evening elongation at ♈ 10° and a morning elongation at ♎ 10°. We can check Ptolemy's results against the correct elongations shown in the table for his dates according to modern theory. Since Ptolemy believes that the apsidal line passes through ♈ 10° and ♎ 10°, we shall do just as he did, and double the one visible elongation in each position. We then have for 1–4, 2–3, 5, and 6,

♒ 10°	$\eta_1 + \eta_4 = 45;1°$,	♊ 10°	$\eta_2 + \eta_3 = 45;27°$,
♈ 10°	$2\eta_6 = 44;22°$,	♎ 10°	$2\eta_5 = 39;10°$.

From this limited information, Ptolemy's conclusion is correct, the sum of opposite elongations is least at apogee in Libra, greater at 180° from apogee in Aries, but greatest ±120° from apogee in Aquarius and Gemini. Hence, one need not appeal to observational errors in the elongations in Aquarius and Gemini to explain the foundation of Ptolemy's theory of Mercury — although there certainly are errors — rather, it was the misfortune of Mercury's invisibility as a morning star in Aries where Ptolemy believed the apsidal line to lie.[12]

Were it possible to observe the morning elongation in Aries and the evening elongation in Libra, Ptolemy's errors, both in the elongations and the location of the apsidal line, would be apparent, as can be shown by computation. The invisible M elongation 6′ to be paired with E elongation 6 occurred 4 years earlier on 131 Apr 4 when $\eta_6' = -25;0°$, and the invisible E elongation 5′ to be paired with M elongation 5 was 4 years later on 138 Oct 4 when $\eta_5' = 21;49°$. That these differ from 5 and 6 shows that the apsidal line does not pass through ♈ 10° and ♎ 10°. In any case, we now have

♒ 10°	$\eta_1 + \eta_4 = 45;1°$,	♊ 10°	$\eta_2 + \eta_3 = 45;27°$,
♈ 10°	$\eta_6 + \eta_6' = 47;11°$,	♎ 10°	$\eta_5 + \eta_5' = 41;24°$,

and it is obvious that the sum of 6–6′ is really much greater than 1–4 and 2–3 while, as expected, 5–5′ is much smaller. Hence, a further misfortune is that the E elongation visible in Aries is so much smaller than the invisible M elongation, which surely also contributed to the mislocation of the apsidal line. Interestingly, if we take the sums of the true greatest elongations η_{max}, some a few days away, we find

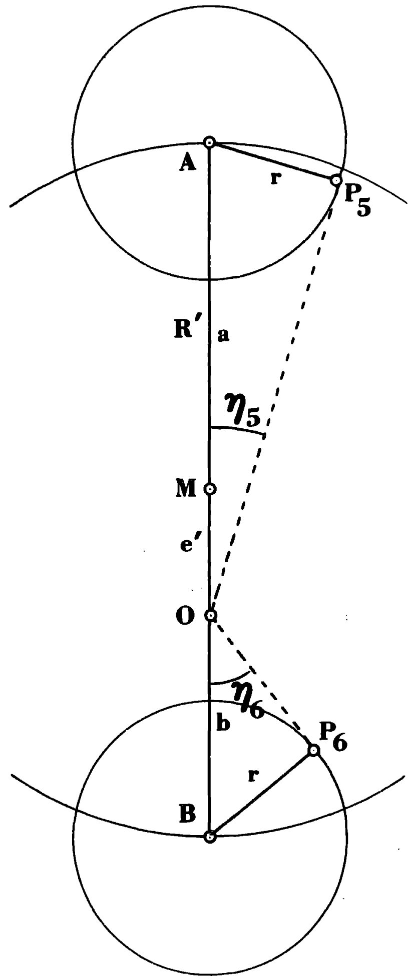

FIG. 8.

♒ 10° $(\eta_1 + \eta_4)_{max} = 45;7°$, ♊ 16° $(\eta_2 + \eta_3)_{max} = 47;13°$,
♈ 10° $(\eta_6 + \eta_6')_{max} = 47;19°$, ♎ 12°–6° $(\eta_5 + \eta_5')_{max} = 42;1°$.

Now 1–4 and 2–3 are nowhere near equal, but 2–3 and 6–6′ are about equal. The reason is that the true apsidal line, found earlier from equal sums of opposite elongations to pass through ♉ 14° ± 4°, is nearly midway between elongations 2–3 at ♊ 16° and 6–6′ at ♈ 10°. And from our recomputation of the elongations during 127–41, we find the sum of opposite elongations a maximum of 47;30° at ♉ 10° and a minimum of 41;20° at ♏ 10–17°, although these cannot be observed because the morning elongation in Taurus and evening elongation in Scorpio are invisible. In the Babylonian theory of Mercury, it is implicit that the planet is invisible as a morning star in Aries and Taurus and as an evening star in Libra and Scorpio, and these missing phases of Mercury were known to Greek astronomy long before Ptolemy, for Ptolemy's demonstration in 13.8 that his theory of Mercury accounts for them implies that they are recognized phenomena.

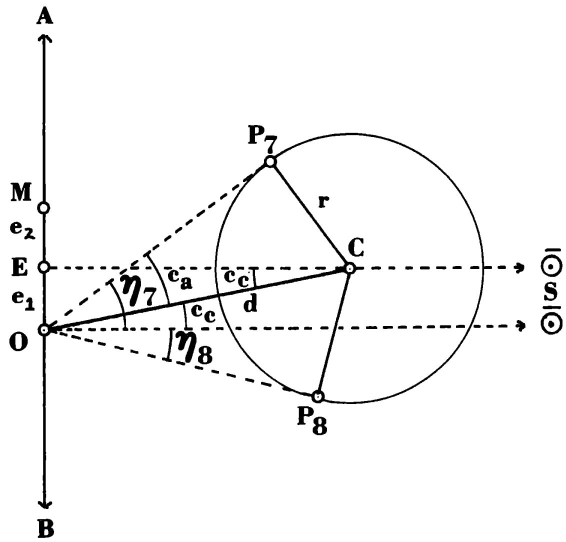

FIG. 9.

The following demonstrations, of the radius of the epicycle and the eccentricities determining distance and direction, are similar to those for Venus. A surprising result, however, is that the eccentricity determining direction turns out to be half, rather than twice, the eccentricity determining distance. Considering first elongations 5 and 6 in the apsidal line, in Figure 8 letting $a = OA = R' + e'$ and $b = OB = R' - e'$, we have

$$r = a \sin \eta_5 = b \sin \eta_6, \qquad b = a\,(\sin \eta_5/\sin \eta_6),$$
$$R' = \tfrac{1}{2}(a + b), \qquad e' = \tfrac{1}{2}(a - b).$$

From $\eta_5 = 19;3°$ and $\eta_6 = 23;15°$, Ptolemy lets $a = 120$, and with small errors finds

$$r = 39;9, \quad b = 99;9, \quad R' = 109;34, \quad e' = 10;25.$$

Now consider elongations 7 and 8 at $-90°$ from apogee, where the equation of centre is near maximum, in Figure 9. The equation of the anomaly c_a and equation of centre c_c are

$$c_a = \tfrac{1}{2}(\eta_8 + \eta_7), \quad c_c = \tfrac{1}{2}(\eta_8 - \eta_7),$$

so that, letting $d = OC$, the distance of the centre of the epicycle,

$$d = r/\sin c_a, \quad e_1 = d \sin c_c.$$

From $\eta_7 = 26;15°$ and $\eta_8 = 20;15°$, where, as before, $r = 39;9$, Ptolemy finds, again with small errors,

$$c_a = 23;15°, \quad c_c = 3;0°, \quad d = 99;9, \quad e_1 \approx 5;12.$$

Note that $e_1 \approx \tfrac{1}{2}e'$, meaning that the equant centre is at one-half the eccentricity of the centre of distance. Thus, in the figure $OE = \tfrac{1}{2}OM$ and $e_2 = e_1$. But something is paradoxical, namely, that both at 180° from apogee, where $c_a = \eta_6$, and here at 90° from apogee, $c_a = 23;15°$, while one would expect c_a to be larger at 180°. Likewise, the distances to the centre of the epicycle at both 180° and at 90° from apogee are equal, that is $b = d = 99;9$, although one would expect the former to be smaller.

Neither the equal angles nor the equal distances are consistent with motion on a circle, or at least with a circle of fixed centre. Since both the distances

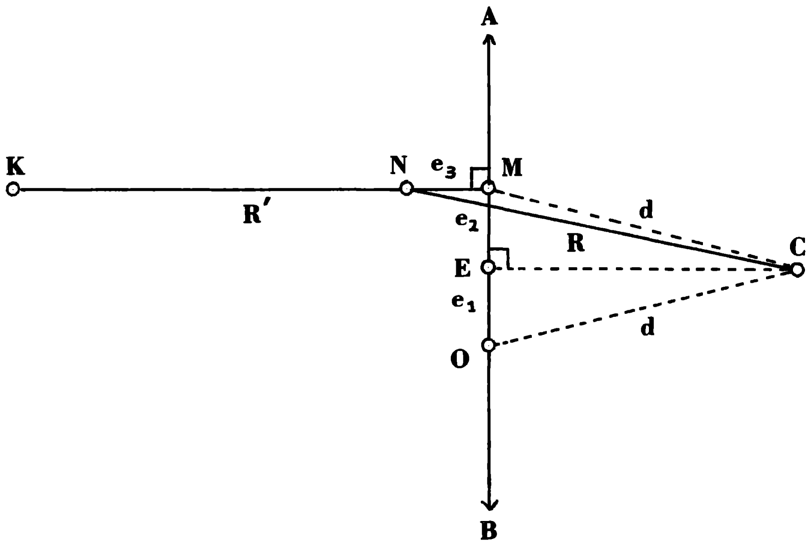

FIG. 10.

and directions of the centre of the epicycle are certain, the problem is this: where at each position of the centre of the epicycle is the centre of the eccentric circle on which it moves? Here is Ptolemy's solution *after* the hypothesis—the motivation for which we shall consider below—has been derived. As shown in Figure 10, when the centre of the epicycle C is $\pm 90°$ from the apogee on line EC, the centre of the eccentric N is $\mp 90°$ from the apogee on line MK, that is, at an equal angle on the other side of the apsidal line. We wish to find the distances $R = NC$ and $e_3 = NM$. Becuase angle $NME = 90°$ and angle $EMC \approx 90°$ (actually $\sim 87°$), the distance $NM + MC$ is nearly a straight line equal to NC, that is, $R \approx d + e_3$. Now, we let $KM = AM = R'$, and since $MC = d$, therefore $NC \approx \frac{1}{2}(KM + MC)$ and $NM \approx \frac{1}{2}(KM - MC)$, that is,

$$R \approx \tfrac{1}{2}(R' + d), \quad e_3 \approx \tfrac{1}{2}(R' - d).$$

It has been found that $R' = 109;34$ and $d = 99;9$, and thus

$$R \approx 104;22, \quad e_3 \approx 5;12.$$

But we earlier found that $e_1 = e_2 \approx 5;12$, and thus all three eccentricities, which we now call e, are equal. Finally, we change to units of $R = 60$ by dividing R, r, and e by $104;22/60 = 1;44,22$ to find

$$R = 60, \quad e \approx 3;0, \quad r = 22;30.$$

The hypothesis that underlies the preceding demonstrations is shown in Figure 11. From the Earth at O, an eccentricity e defines the equant centre E and a double eccentricity $2e$ the centre M of a small circle, also of radius e, that therefore passes through E. Now, the centre of the eccentric N moves in the negative direction, of decreasing longitude, on the small circle through the mean eccentric anomaly $-\bar{\kappa}$ while the centre of the epicycle C moves in the positive direction, of increasing longitude, on the eccentric uniformly through $+\bar{\kappa}$ with respect to E, both measured from the direction of the apogee. We now test the hypothesis and the parameters just derived against Ptolemy's observations. For apogee and perigee, $\eta = c_a = \sin^{-1}(r/OC)$:

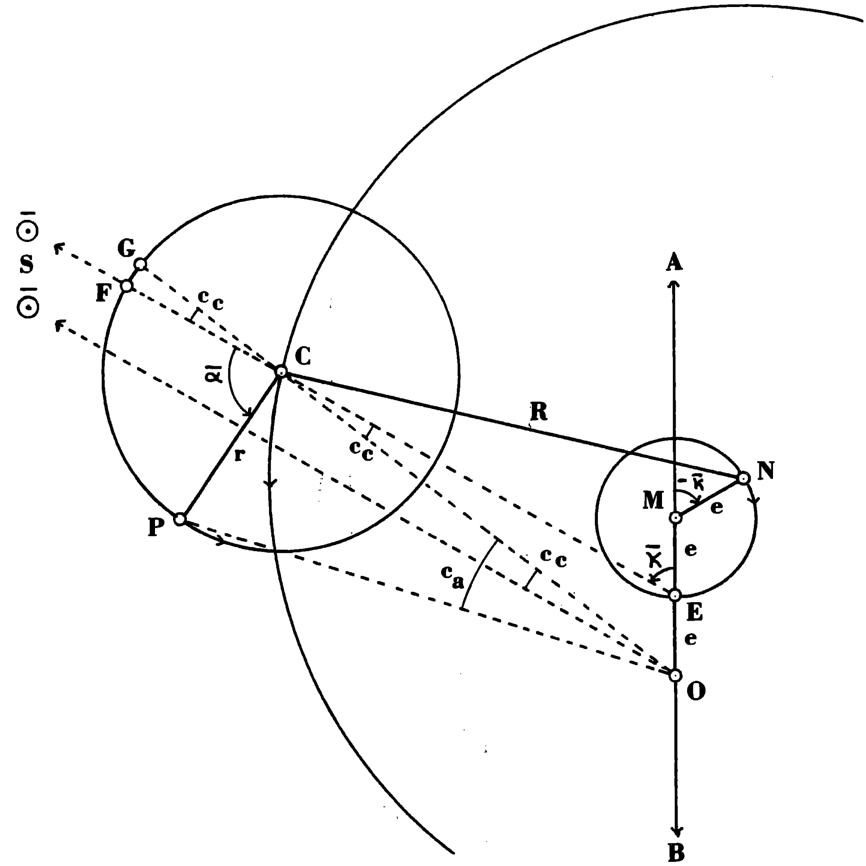

FIG. 11.

$\bar{\kappa} = 0°$: $OC = R + 3e = 69$, $\eta = 19;2°$, obs. $\eta_5 = 19;3°$.
$\bar{\kappa} = 180°$: $OC = R - e = 57$, $\eta = 23;15°$, obs. $\eta_6 = 23;15°$.

For $\bar{\kappa} = \pm 90°$ we must also find the equation of centre c_c, and then $\eta = c_a \pm c_c$:

$\bar{\kappa} = -90°$: $OC \approx RE - e = 57$, $c_c = 3;1°$, $c_a = 23;15°$,
$\eta_E = 26;16°$, obs. $\eta_7 = 26;15°$, $\eta_M = 20;14°$, obs. $\eta_8 = 20;15°$.

The sum of the opposite elongations, $\eta_E + \eta_M = 2c = 46;30°$, agrees exactly with the observed $\eta_7 + \eta_8$.

The calculation for $\bar{\kappa} = \pm 120°$ where the sum of opposite elongations is a maximum, shown in Figure 12 for $+120°$, is more complicated and Ptolemy goes through it in detail. The test is very important for the elongations at $\pm 120°$ have not been used in any of the preceding derivations of eccentricities, and thus if they are correctly produced, the hypothesis must be judged a great success. Ptolemy first shows that at $\bar{\kappa} = \pm 120°$, radius $NC = R$ passes through E and forms in the small circle an equilateral triangle MNE of sides e. This is a fundamental property — indeed, the fundamental property — of the hypothesis regardless of the radius of the small circle, and, as we shall see, is essential for understanding its original motivation. As a result, $EC = R - e$ and:

$\bar{\kappa} = \pm 120°$: $OC = 55;33,38 \approx 55;34$, $c_c = 2;41°$, $c_a = 23;53°$,
$\eta_E = 26;34°$, obs. $\eta_3 = 26;30°$, $\eta_M = 21;12°$, obs. $\eta_2 = 21;15°$.

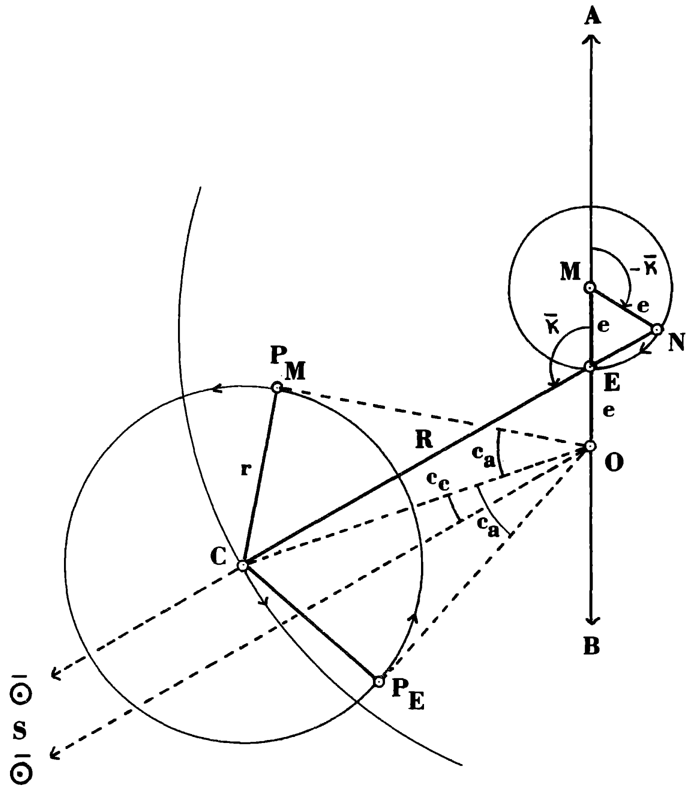

FIG. 12.

The discrepancies of 0;4° and 0;3° are remarkably small, and the sum of the opposite elongations, $\eta_E + \eta_M = 2c_a = 47;46°$, is nearly equal to the observed $\eta_3 + \eta_2 = 47;45°$. Since these elongations were not used in deriving the eccentricities, the hypothesis, by what appears to be sheer luck, works very well indeed.

Or was it really luck? We have thus far reviewed Ptolemy's exposition of the derivation of Mercury's parameters, which is also to some degree a derivation or confirmation of its hypothesis. However, it is not necessarily the way he arrived at the hypothesis or parameters in the first place, as the fortunate outcome of the previous test would seem to suggest. Let us therefore consider another approach. We remarked earlier that a pre-Ptolemaic value of 22° for Mercury's greatest elongation gives Ptolemy's radius of the epicycle, that is, where $R = 60$, $r = R \sin 22° = 22;28,35 \approx 22;30$. If r is taken as known, then the greatest elongations η_E and η_M immediately give the equation of the anomaly $c_a = \frac{1}{2}(\eta_E + \eta_M)$ and the distance of the centre of the epicycle $OC = r/\sin c_a$ where $R = 60$. Hence, for the four positions of the epicycle, expressed as $\bar{\kappa}$, we have:

$\bar{\kappa}$	$\eta_E + \eta_M$	c_a	OC	
0°	2 · 19; 3°	19; 3°	68;56 ≈ 69	*a*
90	46;30	23;15	57	*d*
120	47;45	23;52,30	55;35	
180	2 · 23;15	23;15	57	*b*

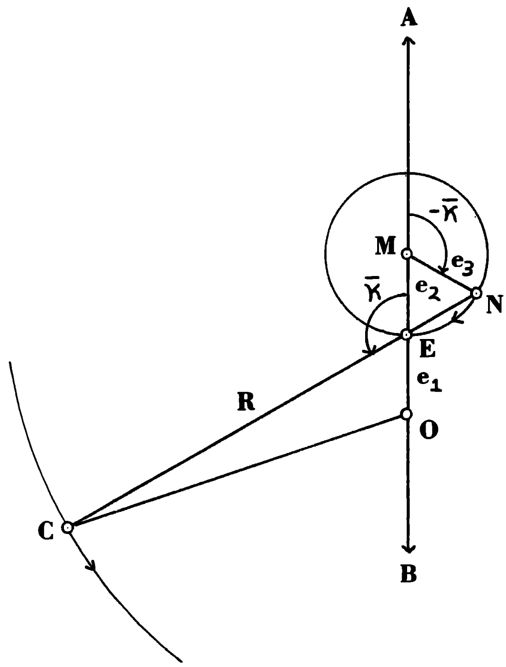

FIG. 13.

Not surprisingly, we have reached almost exactly the distances of Ptolemy's hypothesis and parameters. Assuming that the observed elongations were more or less as reported — a question still to be considered — it is evident that a prior assumption of the radius of the epicycle provides a great advantage in an initial analysis.

The distances, now known, are quite suggestive. We have already drawn attention to the equilateral triangle in the small circle at $\bar{\kappa} = \pm 120°$ regardless of the radius of the circle. This cannot be an accident, but, as we have noted, is the fundamental property upon which the hypothesis was designed in the first place in order to account for the maximum sum of elongations at or near $\bar{\kappa} = \pm 120°$ by bringing the epicycle closer to the Earth. And from the equilateral triangle and the distance at $\bar{\kappa} = \pm 120°$, the radius of the small circle may be derived, so it no longer appears so lucky that the hypothesis produces the correct elongations at $\bar{\kappa} = \pm 120°$. Let us first briefly review the derivations of e_1 and e_2 of Figures 8–9, but now in the new units where $R = 60$. At $\bar{\kappa} = 0°$ and $\bar{\kappa} = 180°$,

$$a = 69, \quad b = 57, \quad R' = \tfrac{1}{2}(a + b) = 63, \quad e' = \tfrac{1}{2}(a - b) = 6.$$

Next, at $\bar{\kappa} = -90°$, as before $c_a = 23;15°$ and $c_c = 3;0°$, so that

$$d = 57, \quad e_1 = d \sin c_c \approx 3 = \tfrac{1}{2}e', \quad e_2 = e' - e_1 = 3.$$

Now consider the configuration for $\bar{\kappa} = 120°$ in Figure 13 in which $NC = R = 60$, but neither the radius of the small circle nor the eccentricity of its centre are yet known; that is, it is not yet known whether M in Figure 13 coincides with M in Figures 8–9. In triangle OEC, from $OC = 55;35$ and $OE = e_1 = 3$, we find

$$EC = (OC^2 + OE^2 - 2OC \cdot OE \cos 120°)^{\frac{1}{2}} = 57;7,32 \approx 57,$$

and in equilateral triangle NME,

$$NE = R - EC = 3, \quad e_2 = e_3 = 3.$$

But it has already been shown that $e_1 = e_2 = 3$, and thus all three eccentricities are equal.

This reconstruction of the preliminary analysis underlying Ptolemy's demonstrations accounts, not only for all the parameters, but also for the lucky coincidence that the hypothesis works so well at $\bar{\kappa} = \pm 120°$. But it does not by itself explain the invention of the hypothesis. Since, as we have seen, the prior assumption of the radius of the epicycle along with the observed elongations gives the distances of the centre of the epicycle, Ptolemy's essential task was to develop an hypothesis to account for the distances, specifically, the smaller distances at, or perhaps originally just near, $\pm 120°$ rather than 180°. The problem is similar to the second inequality of the lunar theory in which the centre of the epicycle is farthest from the Earth at conjunction and opposition, and closest at mean quadrature, that is, at $\pm 90°$ of mean elongation. If, as it appears, the hypothesis of Mercury in the *Canobic inscription* did not differ from that of the other planets, then the hypothesis in the *Almagest* was developed after the lunar theory, which therefore served as an example for such a variation of distance.

In working out the hypothesis, I assume that it was precisely the equilateral triangle in the small circle at $\pm 120°$ that—when he thought of it—convinced Ptolemy that his hypothesis was correct and further, that the maximum sums of elongations really must take place at $\pm 120°$, however ambiguous the observational evidence may originally have been. A necessary condition for the hypothesis is therefore (Figure 13) that the small circle pass through E—meaning that the radius of the circle e_3 must be equal to e_2, which was independently found (Figures 8–9) from $e_2 = e' - e_1$, although not necessarily to e_1—for otherwise NC and EC cannot coincide except in the apsidal line and there can be no equilateral triangle. And this in turn requires that $e' = (e_1 + e_2) > e_1$, that is, that $OM > OE$, which is why for Mercury, and Mercury alone, the equant centre must lie closer to the Earth than the centre of distance. These are very tight constraints on the eccentricities. We shall presently see what happens if these conditions are violated.

It has often been pointed out that the minimum distance from the Earth computed from the hypothesis does not take place exactly at $\pm 120°$, but at about $\pm 120;28,20°$, which is still very, very close.[13] In fact, exactly where the minimum occurs depends upon the eccentricities, something Ptolemy probably knew, for in the later *Planetary hypotheses*, where $e_1 = 3$ and $e_2 = e_3 = 2;30$, it is at about $\pm 124;6,5°$, a notable change of 4°.[14] Hence, probably even in his original analysis, it was not the *exact* locations of the maximum elongations, which observations could not distinguish that closely in any case, but the fact that they were close to $\pm 120°$ that led Ptolemy to develop the hypothesis in which, perhaps because of its elegance, the coincidence of NC and EC and the equilateral triangle in the small circle at $\pm 120°$ determined, not only the hypothesis, but also his interpretation of exactly where the maximum sums of elongations are located. Ptolemy was empirical up to a point, but where observation was inadequate, and in precisely locating the maximum sums of elongations it doubtless was, he had no alternative but to turn to theory, adopting what seemed to him the most simple, reasonable or elegant hypothesis (whatever such criteria may mean) that made sense out of less than certain observations. One must admit that the hypothesis for Mercury is nothing if not ingenious, and Ptolemy himself must have considered it the solution to a problem of truly formidable difficulty.

What then of the empirical foundation of the demonstrations, the observations? Were they altered, corrected, in light of the completed theory? I believe that some, perhaps most, were. This was not done by computation for the time of the observations, for as the table of the observations shows, the planet was not then at true greatest elongation. Rather, the correction was a sort of idealization, as we saw earlier for Venus; that is, *if* the planet were at true greatest elongation at the specific value of $\bar{\kappa}$, then its theoretical elongation is known and the observation can be adjusted accordingly. And again, this explains why Ptolemy always calls the observations "greatest elongations" even though at the time of each observation the elongation was less than maximum. We would probably call this an improper procedure, but I think Ptolemy looked on it more as a kind of interpolation, as a way of making the best out of an unavoidable situation, namely, that the planet is not at greatest elongation when the mean Sun is in the positions required for the demonstrations. In any case, the corrections to the elongations do not affect the fundamental observation that the greatest elongations at $\bar{\kappa} = \pm 120°$ exceed those at 180°, the principle motivation of the hypothesis, for this follows, not from any errors or alterations in the observations, but from the invisibility of Mercury's morning elongations in Aries where Ptolemy believed its apsidal line to be located. For as we have seen, even the correct elongations by modern theory show the same pattern Ptolemy describes: the least morning elongation in Libra, a larger evening elongation in Aries, and a still larger sum of elongations in Aquarius and Gemini, while the largest sum in Taurus and smallest sum in Scorpio cannot be observed owing to visibility conditions.

What Ptolemy observed, and how much the observations were adjusted is a difficult question. If we concede, as I believe we must, that he could not have observed some of the reported elongations — which means that even our reconstruction of his preliminary analysis, based upon corrected elongations, is not exactly preliminary — he still had to start somewhere. It would be interesting if the true greatest elongations, η_{max} in the table, agreed more or less with his reports, but except for 2, 7, and 8 they do not. Since the errors of the corrected elongations amount to as much as $1\frac{1}{2}°$ and several are about 1°, a reasonable guess might be that Ptolemy's adjustments of whatever he observed were of the order of 1°, leaving residual errors in the observations of $\frac{1}{2}°$ or less. But precisely what was the motivation and effect of such corrections? One reason of course is the adjustment to the theoretical true greatest elongation at each location of the centre of the epicycle. But this cannot be done until the theoretical values are known, which themselves depend upon the correction. In fact there is more to the adjustments, and it is perhaps the most interesting consequence of the hypothesis for Mercury. Since the hypothesis, derived from the pattern of the elongations, must be prior to the corrections, and since the radius of the epicycle depends upon its distance — and seems already to have been provisionally established — the corrections must affect principally the eccentricities that determine distance. And what motivates, indeed, determines the corrections are the constraints imposed upon the eccentricities by the hypothesis, namely, that $e_3 = e_2$ and $e' = (e_1 + e_2) > e_1$, without which *NC* and *EC* cannot coincide and form the equilateral triangle *NME* in the small circle.

Suppose we repeat the preliminary analysis using the true greatest elongations, η_{max} in the table (taking 2–3 for $\bar{\kappa} = 120°$), and letting $r = 22;30$, just as we did for the corrected elongations and just as Ptolemy may once have done. The elongations and distances are:

$\bar{\kappa}$	$\eta_E + \eta_M$	c_a	OC	
0°	2 · 20;13°	20;13°	65;7	*a*
90	46;34	23;17	56;55	*d*
120	47;13	23;36,30	56;11	
180	2 · 22;19	22;19	59;15	*b*

Note that $\eta_E + \eta_M$ is greater at 120° than at 180°, but it is also greater at 90°, which has curious consequences. At $\bar{\kappa} = 90°$, $c_c = 2;58°$. From these distances and c_3, we find

$$R' = \tfrac{1}{2}(a + b) = 62;11, \qquad e' = \tfrac{1}{2}(a - b) = 2;56,$$
$$e_1 = d \sin c_c = 2;57, \qquad e_2 = e' - e_1 = -0;1.$$

What this result, that e_2 is negative, implies is that in Figure 13 $OM < OE$ and thus *M* would lie between *O* and *E*, which makes it impossible for *NC* and *EC* to coincide and violates the constraints of the hypothesis. There is no point in continuing the demonstration, for there can be no equilateral triangle in the small circle. We can perform the same derivation using the elongations from modern theory for the dates of Ptolemy's observations. We get a bit farther, for $e' \approx 3;46$, $e_1 \approx 3;14$, and $e_2 = 0;32$, so $OM > OE$. However, taking the configuration for $\bar{\kappa} = 120°$ in the small circle, we find $OC = 58;15$, $EC = 59;56$, and $e_3 = 0;4$, which differs from $e_2 = 0;32$ just found. So this too is impossible. With no other evidence to go by — and Ptolemy had no evidence beside the consistency of this demonstration — these two failures may well be interpreted as proof that the observed elongations are faulty and in need of correction. Ptolemy's choice was therefore to accuse either the hypothesis or the accuracy of the elongations, for both could not be right. And since the hypothesis represented the distribution of the elongations correctly — as indeed it was invented to account for this very distribution — it is more than understandable that, rather than reject the otherwise admirable, and very ingenious, hypothesis, he would decide that the observed elongations contained errors that required corrections.

It therefore appears from this analysis that Ptolemy's adjustments of the elongations were made under the constraints of the hypothesis for the purpose of making the demonstration of the eccentricities work consistently with the

TABLE 4.

No.	Date	$\bar{\lambda}_{\odot}$	Obs η	Com η	Mod η	Date	η_{max}
1.	−261 Feb 12 M	318;10°	−25;50°	−26;21°	−25;23°	Feb 15	−25;40°
2.	−261 Apr 25 E	29;30	24;10	23;57	23;46	Apr 28	24; 1
3.	−256 May 28 E	62;50	26;30	25;59	25;29	May 26	25;33
4.	−261 Aug 23 E	147;50	21;40	22;20	23;57	Aug 25	24; 3
5.	−236 Oct 30 M	215;10	−21; 0	−20;11	−20;41	Oct 27	−21; 4
6.	−244 Nov 19 M	234;50	−22;30	−22; 4	−21;59	Nov 20	−22; 3

hypothesis, the only check he had on the accuracy of the elongations. And further, it is evident where he began, for from the true greatest elongations η_{max} at $\bar{\kappa} = -90°$ in the table, $\eta_{7max} = 26;15°$ and $\eta_{8max} = -20;19°$, nearly identical to Ptolemy's reports, it follows that $c_a \approx 23;15°$ and $c_c \approx 3°$, from which $OC \approx 57$ and $e_1 \approx 3$, all of which agree with Ptolemy's demonstration. In addition, just as was done for Venus, the precise location of the apsidal line was taken as 90° for elongations 7 and 8, and this along with the hypothesis gave the precise locations of the maximum sums of elongations $\pm 120°$ from the apogee. The adjustments of the remaining elongations were directed at finding $e' > e_1$ and $e_3 = e_2$, where $e_2 = e' - e_1$. Since the observations were made near the horizon with unfavourable conditions for accurate measurement and uncertainty about the longitudes of the, sometimes quite distant, reference stars — all of which was known to Ptolemy — adjustments of about 1° to conform with the constraints of a hypothesis that correctly accounted for the distribution and relative sizes of the elongations may not have seemed unreasonable. Indeed, it may have seemed, not only correct, but inevitable. Perhaps the analogy of the two equal eccentricities for each of the other planets led him to conclude that ideally e_2 and e_3 were both equal to $e_1 = 3$, that all three eccentricities were equal, and once this decision was made the adjustment of the elongations was straightforward. This explains in particular the coincidence that $\eta_6 = \frac{1}{2}(\eta_7 + \eta_8)$, for if $e_1 = e_2 = e_3$, then at both $\bar{\kappa} = \pm 90°$ and $\bar{\kappa} = 180°$, the distance $OC \approx R - e_1$, and also why Ptolemy chose to demonstrate e_3 from the distance at $\bar{\kappa} = -90°$ rather than, as one might expect, $\bar{\kappa} = \pm 120°$. Later, in the *Planetary hypotheses* Ptolemy reduced e_2 and e_3 — they must be equal — to 2;30 while leaving e_1 at 3, that is, the eccentricity that can be derived from elongations 7 and 8 at $\bar{\kappa} = -90°$ without any adjustment. The alteration of e_2 and e_3 may have depended upon further observations — Ptolemy says that corrections to the *Almagest* are based upon continuous observation — or a new study and refined adjustment of the original observations.

We may now take up a difficult section of Ptolemy's treatment of Mercury previously deferred, namely, the demonstration in 9.7 from ancient observations that 400 years earlier the apsidal line passed through ♎ 6° and ♈ 6°, 4° less than in his own time. The method is essentially the same as that used before, but since Ptolemy does not have two pairs of equal opposite elongations, he uses six observations, finding the locations of equal elongations by interpolation. The observations are all of distances of Mercury from fixed stars: 1–4, dated in the Calendar of Dionysius, were presumably made in Alexandria, and 5–6 dated in the "Chaldaean Calendar", that is, in the Seleucid Era (−310 Apr) with Macedonian month names substituted for Babylonian, were made in Babylon. The observations are shown in Table 4 arranged as the preceding tables of observations of Venus and Mercury. The observations themselves raise interesting questions. All are about as close to true greatest elongation as Ptolemy's own, but how did Ptolemy know this? Did he determine it by computation or were they already designated as such? Three are from Era Dionysius 23/4 (−261), which suggests either considerable observation of Mercury at that time or a specific interest in greatest elongations. The Babylonian observations, however, were of distances from "normal stars" and could not have specified greatest elongation, so Ptolemy must have gone through some larger list, computing the elongations from the mean Sun and then selecting these two. My guess is that he did the same with the Alexandrian observations which likewise were not specifically of greatest elongations.

The observations are used in two groups, 1–3 and 4–6. In each Ptolemy takes the first elongation directly, and interpolates between the next two to find the location of an equal, opposite elongation. The apsidal line then lies midway between the first and the interpolated location. Thus, taking 1–3, we wish to find the location of an elongation η_1' equal and opposite to $\eta_1 = -25\tfrac{5}{6}°$. Now,

$$\Delta\eta_{32} = \eta_3 - \eta_2 = 2\tfrac{1}{3}°, \quad \Delta\eta_{12} = -\eta_1 - \eta_2 = 1\tfrac{2}{3}°,$$

$$\Delta\bar{\lambda}_{\odot 32} = \bar{\lambda}_{\odot 3} - \bar{\lambda}_{\odot 2} = 33\tfrac{1}{3}°,$$

and by linear interpolation,

$$\Delta\bar{\lambda}_{\odot 21'} = \frac{\Delta\eta_{12}}{\Delta\eta_{32}} \cdot \Delta\bar{\lambda}_{\odot 32} = \frac{5}{7} \cdot 33\tfrac{1}{3}° = 23;48,34° \approx 24°.$$

The location of the elongation η_1' equal to η_1 is

$$\bar{\lambda}_{\odot 1'} = \bar{\lambda}_{\odot 2} + \Delta\bar{\lambda}_{\odot 21'} = ♈\ 29\tfrac{1}{2}° + 24° = ♉\ 23\tfrac{1}{2}°,$$

and since $\bar{\lambda}_{\odot 1}$ = ♒ $18\tfrac{1}{6}°$, the point midway between is at ♈ $5\tfrac{5}{6}°$. Next, we consider η_4' equal and opposite to $\eta_4 = 21\tfrac{2}{3}°$ and interpolate between η_5 and η_6, that is,

$$\Delta\eta_{65} = 1\tfrac{1}{2}°, \quad \Delta\eta_{45} = \tfrac{2}{3}°, \quad \Delta\bar{\lambda}_{\odot 65} = 19\tfrac{2}{3}°,$$

$$\Delta\bar{\lambda}_{\odot 54'} = \frac{\Delta\eta_{45}}{\Delta\eta_{65}} \cdot \Delta\bar{\lambda}_{\odot 65} = \frac{4}{9} \cdot 19\tfrac{2}{3}° = 8;44,26° \approx 9°,$$

and the location of η_4' equal to η_4 is

$$\bar{\lambda}_{\odot 4'} = \bar{\lambda}_{\odot 5} + \Delta\bar{\lambda}_{\odot 54'} = ♏\ 5\tfrac{1}{6}° + 9° = ♏\ 14\tfrac{1}{6}°.$$

Since $\bar{\lambda}_{\odot 4}$ = ♌ $27\tfrac{5}{6}°$, the point midway between is ♎ 6°, and since the other elongations showed ♈ $5\tfrac{5}{6}°$ ≈ ♈ 6°, the apsidal line is directed to ♎ 6° and ♈ 6°. The earliest of these observations (1) is in Era Nabonassar 486 and the latest of Ptolemy's observations (4) is from Era Nabonassar 888, an interval of just over 400 years in which the apsidal line moved 4° from ♎ 6° to ♎ 10°, or 1° in 100 years, which is exactly the rate of the motion of the fixed stars. Hence the apsidal line of Mercury is sidereally fixed and, Ptolemy concludes, since "the phenomena associated with the other planets individually fit" this assumption, the apsidal lines of all the planets are taken to be sidereally fixed.

This demonstration presents grave difficulties, for it is entirely invalidated by errors in the observations, the most serious of which are about +1° in 3 and no less than $-2\frac{1}{3}°$ in 4, which are fatal. Considering absolute values: (1) the interpolation between 2 and 3 is valid only if $\eta_3 > \eta_1$, but in fact they are about equal and by computation from Ptolemy's theory $\eta_1 > \eta_3$; (2) the interpolation between 5 and 6 is valid only if $\eta_6 > \eta_4$, but in fact $\eta_4 >> \eta_6$ and by computation from Ptolemy's theory they are about equal. There is no solution to these problems; the demonstration is invalid and that is that. Elongation 4 is particularly disturbing. The report is that Mercury "was a little more than 3° in advance [west] of Spica, according to Hipparchus's reckoning", meaning that Ptolemy depended upon Hipparchus's reduction, and indeed all the observations probably came from Hipparchus's compilation of earlier observations. For 400 years before his time, Ptolemy takes Spica to be at ♍ $22\frac{2}{3}°$, and subtracts $3\frac{1}{6}°$ to place Mercury at ♍ $19\frac{1}{2}°$. Correctly, by modern theory, the longitude of Spica was ♍ 22;20° and of Mercury ♍ 21;14°, so Mercury was only 1;6° west of Spica, nowhere near 3°. Further, Ptolemy computes the mean Sun to be at ♌ $27\frac{5}{6}°$ while correctly it was at ♌ 27;17°, and the correct elongation was thus 23;57° instead of Ptolemy's $21\frac{2}{3}°$, a difference of nearly $2\frac{1}{3}°$. That these inaccuracies lead to the sensible and consistent conclusion that the apsidal line has moved 4° in 400 years, exactly the motion of the fixed stars, suggests that some of the observations have been adjusted in accordance with this assumption.

But a further problem with Ptolemy's demonstration, obviously enough, is that since he has an error of about −40° in the direction of the apsidal line at his own time, it is difficult to attach much significance to a difference of 4° in 400 years. And to make matters even more confusing, it appears as though in the *Canobic inscription* he had placed the apogee in his own time at ♎ 6°, the very location he gives it here for 400 years earlier.[16] And yet, despite all these problems, his conclusion, that the apsidal line is sidereally fixed, is characteristically the best guess that could have been made at his time even with more accurate observations. For in fact, in 400 years Mercury's apsidal line moves 6;12° with respect to the equinoxes of which 5;32° is due to the precession and only 0;40° a movement with respect to the fixed stars, which could not possibly be distinguished by interpolation for the locations of equal and opposite elongations.

Ptolemy essentially had only two choices with regard to the motion of apsidal lines, whether they were sidereally or tropically fixed, for this was surely the only question he asked and the only question he could hope to answer. The proper motions of the planetary apsidal lines with respect to the fixed stars are far too slow to be detected from only a few hundred years of observations, even suitable and accurate observations which Ptolemy did not have, and Copernicus's attempts of fourteen hundred years later are still hopelessly flawed by errors of observation both ancient and modern. Ptolemy doubtless used the early elongations of Mercury because they were the only early observations he had that could be applied to the problem at all. For the superior planets oppositions would be required in order to derive the directions of the apsidal lines at the early epoch by the same demonstrations Ptolemy used for his own time. But since he does not use early observations of oppositions, where they would be the most appropriate, for correcting the mean motions of the superior planets, it is certain that he had no such early observations. And for Venus he evidently lacked early observations of anything even close to symmetrical greatest elongations, which is understandable considering the unfavourable distribution of Venus's greatest elongations. Hence, under such unsatisfactory conditions, on the basis of little more than inaccurate observations of the least visible planet he made the best guess he could. That is, if all that could be done was to distinguish whether the apsidal lines were sidereally or tropically fixed, then Ptolemy evidently made the correct choice.

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