## The SCAdian Astronomical Epoch

by
Jon Chesey

## Jon Chesey

VoijaRisa@gmail.com
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#### Abstract

The astronomical models of Ptolemy in the $2^{\text {nd }}$ century CE set the stage for how astronomers would predict the motions of the sun, moon, and planets for the next 1500 years. Despite producing accurate results for its period, Ptolemy's astronomical models became obsolete over time due to minor errors in the base variables, astronomical effects not known to him, and ultimately a model that was fundamentally wrong. Accordingly, using Ptolemy's original tables and models today would result in wildly inaccurate results. Thus, attempting to understand how Ptolemy's models would be used by astronomers in period becomes difficult as we cannot directly apply them to present day.

In response, this paper explores Ptolemy's methods for the development of his model and reproduces them using observations from the present, thereby bringing the Ptolemaic model fully into the modern era. To do so, I walk through Ptolemy's methodology, adjusting the geometry and using more expedient modern math where necessary, in order to define a new astronomical epoch. This is a date and time for which the positions of the sun, moon, and planets are all determined and from which, positions in the past or future can then be calculated. In other words, it is $t=0$. In addition, I have recomputed all necessary tables to use for prediction from the epoch, which are included in this work.


## CHAPTER 1

## Introduction

One of the requirements for making astronomical predictions is a point in time that serves as the starting point, for which the position of the sun, moon, and planets are established ${ }^{1}$. From this time and set of positions, and by knowing the way each object moves, their position can be predicted either forwards and backwards in time. This starting date, in combination with the position of an object, is known in astronomy as the epoch. This concept is still used modern astronomy, although modern astronomers generally use the year 2000 as their epoch date.

In the $2^{\text {nd }}$ century CE, Ptolemy wrote the Almagest. In it, he laid the foundation of the geocentric model that would dominate for over a thousand years. For the beginning of his epoch, he chose the first day of the year in which the Nabonassar reign began (747 BCE) ${ }^{2}$. Ptolemy did not determine the starting position of each body on that date by using historical records from that time, but rather found the position in a time contemporary to himself and then reverse calculated the positions for the start of the epoch.

Ptolemy operated under an assumption inherited from the Platonic philosophers that all objects must move in perfect circles with uniform (i.e., constant) speed. This obviously contradicted simple observation which demonstrates that celestial bodies such as the sun, moon, and planets, do not have a constant speed with respect to the background stars. Indeed, the planets can even appear to change their direction. To explain this, Ptolemy built on the work of Greek astronomers before him, describing a complex system in which these celestial objects were carried on the surface of crystalline spheres rotating within one another at constant speeds. By carefully arranging the placement, size, speed and direction of rotation of these spheres, Ptolemy was able to create a model that mimicked the motion of these bodies.

These models were ultimately built on numerous parameters, observations, and assumptions.

[^0]While Ptolemy does an excellent job of justifying each one, he was still limited by the observational tools of his day. Furthermore, there are also subtle astronomical effects that only become apparent over the long term, of which Ptolemy was entirely unaware. Thus, while Ptolemy's models worked reasonably well during his lifetime, astronomers were already finding problems with his work a few centuries later ${ }^{3}$ as the minor errors compounded over these intervals. To try to make use of his work directly today would return results that are wildly out of line with reality. This makes trying to understand the accuracy of models and how astronomers in period may have made use of them difficult.

To make the models usable in present day, they need to be brought into the present, creating a modern epoch from which we can use the models without them being wildly out of line. To that end, my goal is to create a new epoch suitable for use within the Society for Creative Anachronism (SCA). For its starting point, I have chosen the date the SCA was founded: May 1, 1966.

Before diving into the models, we should fist take some time to explore some of the concepts that will be common in all the models.

### 1.1 On the Celestial Sphere

When observing the sky, it appears as a giant sphere around us with half of that sphere always being obscured by virtue of being beneath the horizon. This is referred to as the celestial sphere ${ }^{4}$ and the stars appear to be fixed on this sphere, rotating with it. However, there are seven objects which appear to move with respect to the stars: the sun, moon, and five naked eye planets. The purpose of the Almagest is to model the movement of each of these objects as well as the stars.

In addition to these physical objects, astronomers also consider several more that are more abstract. The first of these is the celestial poles. These are simply the points about which the celestial sphere seems to rotate. They lie directly above the earth's own poles. Similarly, the celestial sphere also has a celestial equator which lies above earth's as well.

Each of the objects that moves with respect to the stars traces out a path along the celestial sphere. But because the sun is such an important object, its path deserves special attention. It is given a special title and is known as the ecliptic.

Both the celestial equator and ecliptic are a special type of circle known as a great circle because their centers are coincident with the center of the celestial sphere and they have the same diameter. In contrast, we can imagine any number of small circles for which they are a slice of

[^1]the sphere that does not go through the center and are therefore smaller in diameter. Any line of latitude aside from the equator would be an example of a small circle.

The above points and circles on the sphere are all ones that are fixed. However, there are several others that only exist in the context of an observer on earth. The first of these is the zenith, which is a point straight up for any observer. Throughout the course of the day, this point traces a small circle around the celestial sphere and $90^{\circ}$ from that point at any time is the observer's horizon which is a great circle. If we draw another great circle between any point on the horizon and the zenith, this is known as an altitude circle, so named because when measuring the angle of an object above the observer's horizon, known as its altitude, it would be measured along this circle. A special case of an altitude circle is the meridian which is the great circle drawn through the observer's zenith and the points directly north and south on the horizon.

### 1.2 On Celestial Coordinates

The coordinate system used throughout the Almagest to describe the apparent position of objects on the celestial sphere is the ecliptic coordinate system. This system is named because it uses, as a fundamental plane, the ecliptic which is the apparent path the sun takes across the celestial sphere throughout the year. Ecliptic latitude is measured up and down from this line with positive values above the ecliptic (northwards) and negative values below (southwards). Ecliptic longitude is measured left and right along the ecliptic with the position of the sun at the vernal (spring) equinox as $0^{\circ}$ and proceeding counter-clockwise when viewed from above or right to left when viewed from inside the celestial sphere.

In Ptolemy's time, the vernal equinox was located at the beginning of the constellation of Aries. However, due to axial precession, this has changed and today the vernal equinox is located near the beginning of Pisces. As such, the sign the sun is in on a given date has shifted too, making the modern zodiacal constellation in which an object would appear be off by approximately one month. This is summarized in the table below:

| Ecliptic Longitude | Classic Sign | Modern Sign |
| :--- | :--- | :--- |
| $0^{\circ}-30^{\circ}$ | Aries | Pisces |
| $30^{\circ}-60^{\circ}$ | Taurus | Aries |
| $60^{\circ}-90^{\circ}$ | Gemini | Taurus |
| $90^{\circ}-120^{\circ}$ | Cancer | Gemini |
| $120^{\circ}-150^{\circ}$ | Leo | Cancer |
| $150^{\circ}-180^{\circ}$ | Virgo | Leo |
| $180^{\circ}-210^{\circ}$ | Libra | Virgo |


| $210^{\circ}-240^{\circ}$ | Scorpio | Libra |
| :--- | :--- | :--- |
| $240^{\circ}-270^{\circ}$ | Sagittarius | Scorpio |
| $270^{\circ}-300^{\circ}$ | Capricorn | Sagittarius |
| $300^{\circ}-330^{\circ}$ | Aquarius | Capricorn |
| $330^{\circ}-360^{\circ}$ | Pisces | Aquarius |

### 1.3 Eccentric vs Epicyclic Models

In the Almagest, Ptolemy lays out two main models to help account for the irregular motion of celestial objects: the eccentric and the epicyclic models. This is initially done in Book III Chapter 3 (abbreviated III.3) for the sun, wherein Ptolemy demonstrates that the two models produce the same results so long as certain parameters are equal. He later demonstrates this again in Book IV for the moon for the the simple model which he later replaces.

The first of these models is the eccentric model. This model uses a single sphere for the motion of the object which is offset from the earth as shown in Figure 1.1. In this figure, the observer is on earth at $E$ and the sun travels on circle $A P O$ which has center $C$. This offset circle is referred to as an eccentre. This naturally creates a point at which the object would be closest to the earth, known as perigee at $P$, and one at which is most distant, known as apogee at $A$. The effect this has for a viewer on earth is that it causes the apparent motion of the object to vary, seeming to move faster near perigee and more slowly near apogee, while the object maintains a consistent angular speed about this circle if viewed from $C$.

The specifics of how this works are illustrated in Figure 1.1. To understand, consider an object at $O$. From the point of view of the observer on earth, the object's apparent position on the ecliptic is at $O_{1}$.


Figure 1.1: An illustration of the eccentric model as it existed in Ptolemy's time. Not to scale.

Next, consider what would happen if we moved the eccentre such that its center, $C$, was moved to be coincident with the earth and the center of the ecliptic at $E$. If that happened, the object would still maintain its orientation in relation to $C$. I've sketched in a dotted-dashed line from $C$ to $O$ to illustrate that orientation. I've then reproduced it, starting from $E$ to show where the object would instead appear on the ecliptic: at point $O_{2}$.

Since ecliptic longitude is measured counter-clockwise, this means that this offset from the earth makes the object appear ahead of where it should be were the eccentre not offset. The opposite is true for an object on the other side of the line between the apogee and perigee. In this case the object appears to lag behind the position it would were the eccentre centered on the earth. The motion that the object would have were the eccentre not offset is known as the mean (average) motion, and the variance from the mean is known as the anomaly or anomalistic motion.

This cyclic pattern of getting ahead and falling behind the mean motion is used to explain why celestial objects appear to vary their speed while maintaining a constant motion on their own circles. However, it cannot account for more complicated motions such as the apparent reversal in direction known as retrograde motion that planets exhibit.

For such motions, Ptolemy instead relies upon the epicyclic model which can account for the simple anomalistic motion as well. In this model, the object travels on two spheres. The first, which in Figure 1.2 has $M$ on its circumference, is known as a deferent, and it carries a second sphere, shown by circle $A P O$, centered on $M$, and on whose surface the object rests. This second sphere is known as the epicycle. As the epicycle rotates about $M$, carrying an object, it again
naturally creates an apogee and perigee, labeled $A$ and $P$ respectively.


Figure 1.2: An illustration of the epicyclic model.

In this model, both the deferent and epicycle rotate with constant speeds, but the motion of the epicycle causes the objects speed to appear to vary. We can understand why by considering Figure 1.2 when the object is at $O$. In this case, the apparent position of the object along the ecliptic is at $O_{1}$ which lags behind the position the object would appear if there were no epicycle and the object were carried directly on the deferent at $M$. Without the epicycle the object would instead appear at $M_{1}$. The opposite would be true if the epicycle rotated carrying the object to the other side of the epicycle. At apogee and perigee, there is no apparent difference in position due to the epicycle.

As the epicycle rotates, carrying the object on its surface, there are times when the motion of the object will be in the same direction as the deferent, and at other times opposed. For example, if the deferent and epicycle were both rotating counter-clockwise in Figure 1.2, then the epicycle would be making the object appear to move faster near perigee and more slowly near apogee. If the rotation of the epicycle were instead clockwise, the opposite would be true: the epicycle would make the object appear to move more slowly near perigee and faster near apogee.

Throughout the Almagest, Ptolemy uses variations on these models, adjusting parameters of the sizes, rotational speeds and rotational directions of the various circles, as well as occasionally combining the two models for especially challenging motions. However, because the models are comprised of discrete parts, these complex motions can be broken down into simpler motions to make computation far easier.

### 1.4 On Ptolemy's Geometry

In this paper, I have done my best to reproduce Ptolemy's methods, but have simplified the procedures to do so by using modern math where appropriate. One example is that I have displayed all numbers in decimal form as opposed to the base 60 sexigesimal system used by Ptolemy. When doing so, I often give a result that may seem like a ridiculous number of decimals which would certainly exceed the number of significant digits. However, this reflects Ptolemy's values which are frequently expressed to the $60^{-2}$ place (equivalent of $0.0002 \overline{7}$ ). Clearly, the concept of significant figures had not yet been developed as Ptolemy regularly plays fast and loose with the number of places given, frequently resulting in compounding rounding errors in his calculations.

In addition, the field of trigonometry (particularly, $\sin , \cos$, and tan functions) was not invented until several hundred years after Ptolemy's death. This meant that simple problems of solving right triangles, which we would do using trigonometric functions in a few steps, required Ptolemy to use roundabout geometric solutions. While these techniques are elegant, teaching them so as to be able to reproduce Ptolemy's methods exactly is beyond the scope of this paper. Rather, in this paper, I have used some modern mathematical techniques to more quickly derive the same results.

That being said, there are certain conventions Ptolemy uses which are not so easily hidden. As such, we should briefly explore some of them ${ }^{5}$.

### 1.4.1 Circles

Circles frequently come up when working with the Almagest since Ptolemy considers objects moving on the surfaces of rotating spheres. Thus, the path they make as the sphere spins traces out a circle. Determining the size of these spheres will be a major component of Ptolemy's work. In addition, when solving right triangles, his methodology involves inscribing the triangle inside a circle for which there is no physical analogue. Ptolemy can then use theorems about circles to solve the triangles in the absence of trigonometry.

Since the true size of the elements within the circles is not something Ptolemy starts out knowing, Ptolemy regularly assigns an arbitrary radius to them of $60^{p}$ where the $p$ stands for "parts". These units are only consistent within the context of the circle for which they are defined. However, by finding the length of a segment in two different contexts (i.e., in two differently sized circles), he is able to switch between them. We will see this in particular in the section on the lunar models.

There are a few important mathematical concepts and theorems that will be important to know. The first is the concept of a chord. A chord is a line drawn between two points on the perimeter of

[^2]a circle that cuts through the circle itself. The length of this chord will have a direct relationship to the arc defined by the same two points. When a triangle is inscribed in a circle, it creates three chords, thus allowing Ptolemy to use relationships and theorems about them. In the Almagest, Ptolemy spends Book I, Chapter 10 showing how to calculate the length of a chord for a given central angle (or arc). To avoid having to repeat such calculations every time this relationship is needed, Ptolemy calculates the length of a chord for increments of $\frac{1}{2}^{\circ}$ and records them in a table in Chapter 11. However, it frequently happens that the values that get input into this table are not in increments of $\frac{1}{2}^{\circ}$ requiring interpolation. Thus, for the purposes of this paper, I have avoided making use of these tables and instead determine the chord length from the central angle/arc (or vice versa) using trigonometry.

Second, if a right triangle is inscribed in a circle, its hypotenuse will be the diameter and the center of the hypotenuse the midpoint the center of the circle. Since Ptolemy's circles have a radius of $60^{p}$, this will mean that any circles he creates for the purposes of solving triangles will have a diameter, and thus the hypotenuse of the triangle, of $120^{p}$.

Third, Ptolemy defines the circumference of a circle to have $360^{\circ}$. Never does Ptolemy define the circumference of his circles in terms of parts as he does the chords. This is helpful because it means the arc is equal to the central angle that it subtends allowing us to switch between them fluidly.

Lastly, the angle a chord (or arc) subtends from the center of a circle is always twice the angle if the vertex is drawn on the perimeter of a circle. To understand how this is helpful, consider a side of a triangle inscribed in a circle. The length of that chord will allow us to determine the arc defined by the same endpoints. Since those arcs are defined in degrees, the angle opposite that side in the triangle (which is on the perimeter of the circle) will always have half the measure of the arc defined by the chord, so long as the point is not within the arc created by the points that define the chord.

### 1.5 On Calendars

A brief but important consideration before going forward is to consider how Ptolemy regarded calendars since he did not use the Julian calendar, despite it existing in his time. Instead, Ptolemy relies on the Egyptian calendar. This calendar has twelve months in which each month consists of 30 days. The calendar then has five epagomenal days inserted at the end of the year but are not considered part of any month. This gives a total length for the Egyptian year of 365 days. However, the calendar did not include intercalary days, more commonly known as "leap days."

The reasoning for this is not explained by Ptolemy, but it likely stems from the reason that the Almagest makes heavy use of tables for which the motion of objects on their spheres is given for
various intervals of time, including years. Thus, he requires a year of a fixed duration to correspond to a fixed amount of motion. Otherwise, if the number of days in a year could vary by having leap days, then how would one define the motion in a year?

While Ptolemy does not discuss it, his calculations still make it apparent that he was well aware of the need for leap days. Indeed, if he did not consider them when determining the interval of time between two dates, then there would be days for which the motion of the objects was not accounted and the results of his calculations would quickly become inaccurate. For example, the mean speed of the sun along the ecliptic is nearly $1^{\circ}$ per day. Thus, if Ptolemy failed to account for an intercalary day, then the calculated position of the sun would be off by that amount.

One place we can see that Ptolemy is aware of the need for such days is in the first chapter of Book III where he discusses the length of the year. By comparing the dates of the solstices and equinoxes to records of astronomers several hundred years earlier, he is able to determine the length of the year to be 365 days, 5 hours, 55 minutes, and 12 seconds ( $365.24 \overline{6}$ days). This is an impressively accurate value, only about 6 minutes over the correct one.

Lastly, Ptolemy reckoned the beginning of the day from noon on that day. As such, when his dates are translated to conform to our modern calendar, the dates are frequently given as two dates since one of his days spanned two of ours. Following Ptolemy's definition, I have started the epoch defined in this paper from noon on May 1, 1966.

### 1.6 On Time

One challenge that Ptolemy must frequently wrestle with is that of time zones. While these did not formally exist in his time, Ptolemy makes frequent use of observations from astronomers before him who lived in what we would consider different time zones than his native one of Alexandria in modern day Egypt. Ptolemy adopts his local time as the standard throughout the Almagest, and thus converts all observations to Alexandrian local time. For my updating of Ptolemy's models, I have done likewise, using my home city in the Barony of Three Rivers (mka: St. Louis, MO) as the standard which exists in the Central Time zone. For ease of use, I will always display things in the appropriate time zone for my updating.

However, this does introduce a complication with which Ptolemy did not need to trouble himself: Daylight savings time. This annual adjustment of the clocks means we will need to pay extra careful attention to times, adjusting as necessary so we always have a consistent system. For this paper, I will always use Central Standard Time (CST). Thus, if a date falls during daylight savings time (i.e., is Central Daylight Time, CDT), we will need to subtract an hour to adjust to CST.

## CHAPTER 2

## Solar Model

The first model Ptolemy begins developing is the solar model. This choice is a reasonable one as the solar model will later be used to calibrate the other models requiring that it be developed first. In addition, the sun's motion is relatively simple making it an easy starting point. While it speeds up and slows down, its path across the sky defines the ecliptic which is used as the baseline for the ecliptic coordinate system and astronomers throughout period used. As such, there is no need to worry about its ecliptic latitude. The sun's motion is also simple such that there is no retrograde motion. Because of this simplicity, Ptolemy is free to use either the eccentric or epicyclic model as both can reproduce this simple motion. He ultimately settles on the former, having the sun move counter-clockwise on the eccentre since that is the direction the sun moves through the zodiac signs.

This choice of model requires surprisingly few pieces information to calibrate, although determining them is not always straightforward. They include the mean speed of the sun on the deferent, the location of the center of the eccentre, and a date for which the sun's position was known from which to reverse calculate the epoch.

### 2.1 Solar Mean Motion

The first of these parameters is quite easily to determine. Since the sun travels a full circle around the sky each year, we can determine the daily motion by dividing $360^{\circ}$ by the number of days in a year. Ptolemy used a value of $365.24 \overline{6}$ days. Thus, we get that the sun's mean motion is $0.9856352784 \frac{\circ}{\text { day }}$.

While we could multiply this by any interval of time, expressed in days, to determine how far the sun moved about the eccentre in that interval, Ptolemy makes it easier and performs this calculation for us, giving the motion of the sun in 18 year intervals, single years, Egyptian months, single days, and single hours. This he lays out in III. 2 and as these values have not changed since his time, I have reproduced this table in Appendix B.


Figure 2.1: Diagram for determining the position of the eccentre.

### 2.2 Ecliptic Longitude of Apogee

Next, we can ask where the eccentre is located in relation to the earth and explore the effects it will have on the anomalistic motion. In particular, this location will have two consequences to the anomaly. The first is that the position of the eccentre determines the direction of apogee and perigee which, as we have seen, determines whether the anomaly causes the apparent position of the sun to lag or precede the position the sun would have were the eccentre not offset. Second, the more distant the center of the eccentre is from the earth, the larger the anomalistic effect. We will begin by understanding the location of apogee, expressing it in ecliptic longitude.

To do so, we construct Figure 2.1. In it, circle $A B G D$ is the ecliptic ${ }^{1}$, centered on the Earth at $E$. It is drawn such that $A$ is the vernal equinox, $B$ the summer solstice, $G$ the autumnal equinox, and $D$ as the winter solstice.

The solar eccentre is circle $N P O S$, with center $Z$, offset from the circle of the ecliptic by distance $\overline{E Z}$. The points of apogee and perigee are unlabeled on this circle, but the projection of the apogee onto the ecliptic is labeled at point $H$. We can quickly notice that the arcs of the eccentre are not even in the quadrants defined by the ecliptic. This matches the fact that the length of time is not equal between subsequent equinoxes and solstices. Ptolemy used the ratio of times between each equinox and solstice to a full year in order to determine the proportion of the $360^{\circ}$

[^3]motion in that same time. He describes the method by which he determined these dates in III.1. There, he describes an equatorial armillary which, at its simplest, is a metal ring fixed such that it is perfectly parallel to the earth's equator. So for an observer on the equator, this ring would be fixed perfectly vertically. As latitude increases for the observer, the ring would be tilted lower and lower with an angle from the vertical corresponding to the latitude until the observer was at the poles in which case it would be parallel to the ground.

To understand how this would be used, consider the position of the sun at its extremes in summer and winter. In the summer, the sun is high in the sky. Thus, the shadow from the edge of the ring towards the sun would fall below the back ring. In winter, when the sun is lower in the sky, it would be above the side further from the sun. On the equinoxes, the sun would be between these two extremes and the side of the ring toward the sun would cast a shadow directly on the further side. The date that happened, with a sliver of light changing from the bottom to the top of the far ring, was the autumnal equinox.

Instead of determining this myself, I have used modern sources. In $2000^{2}$ [11], the vernal equinox was on March 20 and happened at 1:30am $\mathrm{CST}^{3}$. At that time, the sun's projection onto the ecliptic would have been at $A$ meaning the sun's true position on the eccentre would be at $M$.

The following summer solstice occurred on June 20 around $8: 00 \mathrm{pm} \mathrm{CST}^{4}$ [11]. Since $B$ is the summer solstice on the ecliptic, this would mean the sun at this time would have been at $\Theta$. The interval between these two dates is 92 days and 18.5 hours ( 92.77 days).

Since the sun moves on the eccentre at a constant rate, the ratio of this interval of time to a year is equal to the ratio it moved on the eccentre to a complete circle:

$$
\frac{92.7708 \overline{3}}{365.24 \overline{6}}=\frac{\operatorname{arc} M \Theta}{360^{\circ}}
$$

Solving, we determine that arc $M \Theta=91.44^{\circ}$.
Similarly, we can determine arc $\Theta K$ using the interval of time between the summer solstice and the autumnal equinox, $G$ on the ecliptic when the sun would have been at $K$ on the eccentre. In 2000, the autumnal equinox occurred September 22 at 11:30am CST $^{5}$ [11]. The interval of time between the summer solstice and this equinox is 93 days and 15.5 hours or $93.6458 \overline{3}$ days, allowing us to write:

[^4]$$
\frac{93.6458 \overline{3}}{365.24 \overline{6}}=\frac{\operatorname{arc} \Theta K}{360^{\circ}}
$$

Solving gives arc $\Theta K=92.30^{\circ}$.
We can repeat this again for arc $K L$, which represents the motion of the sun on the eccentre between the autumnal equinox and the winter solstice which occurred on December 21 at 7:30am $\mathrm{CST}^{6}$. The interval between these two dates is 89 days and 20 hours or $90.8 \overline{3}$. Again writing this out as the ratios:

$$
\frac{89.8 \overline{3}}{365.24 \overline{6}}=\frac{\operatorname{arc} K L}{360^{\circ}}
$$

Solving we find arc $K L=88.54^{\circ}$.
We could repeat this process for arc $L M$ but it is not necessary for the remaining solution. Instead, we can add arc $\Theta K$ to $\operatorname{arc} K L$ to get $\operatorname{arc} \Theta L=180.84^{\circ}$.

We can then observe that $\operatorname{arc} \Theta N=\operatorname{arc} O L$. The sum of these two arcs is $\operatorname{arc} \Theta L-\operatorname{arc} N O$. Next, we can see that $\overline{N O}$ bisects the eccentre and thus, $\operatorname{arc} N O=180^{\circ}$. Thus,

$$
\operatorname{arc} \Theta N+\operatorname{arc} O L=180.84^{\circ}-180^{\circ}=0.84^{\circ}
$$

And since those two arcs are equal in length, each will have a length of half this or $0.42^{\circ}$.
Let us now focus on the top of these two arcs. If we draw a line, extending from point $\Theta$ such that it meets $\overline{Z N}$ perpendicularly at $T$, and connect $Z$ to $\Theta$, we form right triangle $Z \Theta T$. From this, we can determine the length of this new line, $\overline{\Theta T}$, using a bit of trigonometry. In this triangle, $\overline{Z \Theta}=60^{p}$ since it is a radius of the eccentre, the true size of which Ptolemy does not currently know. In addition, $\angle T Z \Theta=0.42^{\circ}$ since it is subtended by arc $\Theta N$. Applying some basic trigonometry:

$$
\sin \left(0.42^{\circ}\right)=\frac{\overline{T \Theta}}{60^{p}}
$$

Solving we determine that $\overline{T \Theta}=0.44^{p}$. Because this line segment is bounded by $\overline{N O}$ and $\overline{B D}$ which are parallel, any other parallel segments bounded by these other two lines will be equal. This means that $\overline{T \Theta}=\overline{Z R}=\overline{X E}=0.44^{p}$.

Let us now turn our attention to $\operatorname{arc} \Theta K$. This ark is composed of three distinct segments: $\operatorname{arc} \Theta N$, arc $N P$, and arc $P K$. The sum of these, as already shown, is $92.30^{\circ}$. In addition, we have demonstrated that $\operatorname{arc} \Theta N=0.42^{\circ}$, and we can see $\operatorname{arc} N P=90^{\circ}$. As such, we can subtract each of those two arcs from $\operatorname{arc} \Theta K$ to determine $\operatorname{arc} P K=1.88^{\circ}$.

Again, we can create a right triangle by extending a line from $K$, perpendicular to $\overline{P Z}$ at point

[^5]$F$ and then connecting $K$ to $Z$. We again know that $\overline{K Z}=60^{p}$ as it's again a radius of the eccentre. In addition, we know that $\angle F Z K=1.88^{\circ}$ since it is equal to the arc subtending it. We can then apply the same trigonometry:
$$
\sin \left(1.88^{\circ}\right)=\frac{\overline{F K}}{60^{p}}
$$

Solving we see that $\overline{F K}=1.97^{p}$.
As with before, $\overline{F K}=\overline{Z X}=\overline{R E}=1.97^{p}$, since they are all bounded by the same parallel lines, $\overline{P R}$ and $\overline{K E}$.

Now let us consider the right triangle $Z E X$. We have succeeded in finding two of the sides in this triangle: $\overline{Z X}$ and $\overline{X E}$ (which were $1.96^{p}$ and $0.44^{p}$ respectively). If we consider $\angle Z E X$ in this triangle, we can see that it is the same angle as $\angle H E G$ which is the angular distance in ecliptic longitude of the apogee, $H$, from the autumnal equinox, $G$. We can solve for this angle with a bit of trigonometry:

$$
\tan \angle Z E X=\frac{\overline{Z X}}{\overline{X E}}=\frac{1.97^{p}}{0.44^{p}}
$$

Solving for $\angle Z E X$ we determine that it is $77.41^{\circ}$ before the autumnal equinox which is $102.59^{\circ}$ ecliptic longitude (since the autumnal equinox is defined to be $180^{\circ}$ in ecliptic longitude). For the modern-day position of the vernal equinox, this places the sun just over $12 \frac{1}{2}^{\circ}$ into Gemini at apogee. As we will see in Chapter 2.3, only at apogee and perigee can we directly relate the position on the eccentre to the position on the ecliptic since only at those two points does the effect of the offset between their respective centers becomes zero.

Before proceeding, let us pause and consider the accuracy of this result. The actual apogee in 2000 was on July 3 at $5: 49 \mathrm{pm} \mathrm{CST}^{7}$ [1]. At that time, Stellarium puts the sun at $102.28^{\circ}$ ecliptic longitude [7] and the International Meteor Organization's (IMOs) Solar Longitude Calculator gives a value of $102.21^{\circ}$ [12]. Thus, our result is about $0.35^{\circ}$ high.

This error is likely due to the rounding done when determining the days in each interval. We can understand why when we consider the small angles that were plugged into the sin function. Because the sin function has its greatest slope when near $0^{\circ}$, this means it is sensitive to small changes. Toomer notes that being even a single hour off could result in an error of $1^{\circ}$ in the position of apogee [10]. Thus, the error here is well within reason.

[^6]
### 2.3 Effect of Eccentricity and the Solar Anomaly

Now that we have determined the direction of apogee, we can next use some of the work to determine the distance between the earth and the center of the eccentre. Again referring to Figure 2.1, we can see that we already have sides $\overline{Z X}$ and $\overline{X E}$ of right triangle $Z X E$ in which $\overline{E Z}$ is the distance between the earth and the center of the eccentre. Thus, we can quickly use the Pythagorean theorem to determine this distance:

$$
1.97^{2}+0.44^{2}=\overline{E Z}^{2}
$$

Solving, we determine $\overline{E Z}=2.02^{p 8}$. With that distance in hand, we can now determine what the effect of this offset will be. Ptolemy explores this in III. 5 in which he determines the deviation from the mean motion (i.e., what the position of the sun would be if the eccentre were centered on the earth) for various points about the eccentre based on their distance from apogee. This deviation from the mean motion is known as the equation of anomaly or more simply the anomaly.


Figure 2.2: Diagram for determining the equation of anomaly based on distance from apogee.

In Figure 2.2, the ecliptic is drawn as circle $A B G$ centered on the earth at $D$. The sun's eccentre is circle $E Z H$ centered on $\Theta$.

As an example, we will consider the case when the sun is at $Z, 30^{\circ}$ from apogee at $E$, meaning $\operatorname{arc} E Z=30^{\circ}$. We will attempt to find $\angle D Z \Theta$ as this is equal to the equation of anomaly as

[^7]Ptolemy demonstrates in III.3. To do so, we have extended $\overline{Z \Theta}$ such that it meets a line extended from $D$ at point $K$, meeting perpendicularly.

Since $\operatorname{arc} E Z=30^{\circ}$, so too is the angle it subtends, $\angle E \Theta Z$ as well as $\angle D \Theta K$ since it is a vertical angle to $\angle E \Theta Z$. We just demonstrated that the distance between the earth and the center of the eccentre was $2.02^{p}$ in the context of the eccentre. Thus, we can apply some trigonometry to $\triangle D \Theta K$ and solve for $\overline{\Theta K}$ :

$$
\cos \left(30^{\circ}\right)=\frac{\overline{\Theta K}}{2.02^{p}}
$$

Solving, we determine $\overline{\Theta K}=1.75^{p}$. We can then quickly use the Pythagorean theorem to determine $\overline{D K}=1.01^{p}$

We can then add $\overline{\Theta K}$ to $\overline{Z \Theta}$ which has a measure of $60^{p}$ since it's the radius of the eccentre, to determine $\overline{Z K}=61.75^{p}$.

Next, consider right triangle $D Z K$. We now know two of the sides in this right triangle and can therefore use trigonometry to determine $\angle D Z \Theta$ :

$$
\tan (\angle D Z \Theta)=\frac{1.01^{p}}{61.75^{p}}
$$

Solving this for $\angle D Z \Theta$, we determine it is equal to $0.94^{\circ 9}$. Thus, when the sun is $30^{\circ}$ from apogee on its eccentre (measured counter-clockwise), it will appear to be $0.94^{\circ}$ away from the position one would expect based on its mean motion alone. Whether it is increased or decreased will be determined by whether it is before or after the apogee. As we saw in 1.3, if the object is before the apogee the effect will be additive. If it is after the apogee, it will be subtractive.

Ptolemy repeats this calculation, for numerous angles from apogee, going in $6^{\circ}$ increments from $0^{\circ}-90^{\circ}$ from apogee, and then in $3^{\circ}$ increments from $90^{\circ}-180^{\circ}$ where the sun will be closer to perigee and thus moving faster giving need for a greater resolution. His results are displayed in III. 6 of the Almagest. I have followed suit, and my results are available in Appendix C.

### 2.4 Determining Epoch Position

The final step before reverse calculating the position of the sun at the beginning of the epoch will be to select a date for which we now know the position of the sun as well as both the sun's mean position and equation of anomaly. It may be tempting to use the apogee for which there is no anomaly, but this date was not known to Ptolemy ${ }^{10}$.

[^8]Instead, we will choose the autumnal equinox since we have already determined the distance of apogee from this solstice (Chapter 2.2). Specifically, we showed that, on the the autumnal equinox, the sun was $77.41^{\circ}$ past apogee (going counter-clockwise) on the ecliptic. However, this position is composed of both the mean position as well as the effect of the anomaly. In other words

$$
\text { Ecliptic longitude }=\text { mean position } \pm \text { equation of anomaly }
$$

So far in this equation, we have determined the ecliptic longitude of the sun on the autumnal equinox, but will need to determine the equation of anomaly in order to solve for the mean position which is the distance of the sun from apogee on the eccentre. From there, we can apply the mean motion of the sun over the interval since the beginning of the epoch to determine the sun's mean position at that time.


Figure 2.3: Diagram for determining the position of the sun on the eccentre for the autumnal equinox in 2000.

To do so, we will make use of Figure $2.3^{11}$. In this diagram circle $E Z$ is the eccentre, centered on $\Theta$. The ecliptic is circle $A B$ centered on the earth at $D$ with point $B$ being the position of the sun as viewed from earth on the autumnal equinox, thus making point $Z$ the position of the sun on the eccentre at that time. To assist us in our math, I have also dropped a line from $\Theta$ onto $\overline{B D}$ such

[^9]that it falls perpendicularly at point $K$.
In Chapter 2.2, we determined that the apparent position of the sun on the ecliptic at apogee, $A$ is $77.41^{\circ}$ from the autumnal equinox which we can now consider to be point $B$, thus making arc $A B=77.41^{\circ}$ as well as the central angle it subtends, $\angle B D A$. If we consider this same angle in $\triangle \Theta K D$, it is $\angle K D \Theta$. Within this triangle we also determined in Chapter 2.3 that $\overline{D \Theta}=2.02^{p}$ (there called $\overline{E Z}$ ) in the context of the eccentre where its radius is $60^{p}$. Since this is a right triangle, we can use trigonometry to solve for $\overline{K \Theta}$.
$$
\sin \left(77.41^{\circ}\right)=\frac{\overline{K \Theta}}{2.02^{p}}
$$

Solving, we find that $\overline{K \Theta}=1.97^{p}$.
We can then consider $\triangle \Theta Z K$, which is also a right triangle. In it, we know $\overline{Z \Theta}=60^{p}$ and $\overline{K \Theta}=1.97^{p}$ so we can again apply some trigonometry to determine $\angle \Theta Z K$ which is equal to the equation of anomaly.

$$
\sin (\Theta Z K)=\frac{1.97^{p}}{60^{p}}
$$

Solving, we see that $\angle \Theta Z K=1.88^{\circ}$. As with before, this is the equation of anomaly. Recalling that the sun moves counter-clockwise on the eccentre, we can see that its motion from $Z$ to $E$ must greater than $180^{\circ}$. As discussed in Chapter 2.2, this means its position would be found in the second column of the Table of Solar Anomaly and thus have a subtractive effect on the mean position. Therefore to cancel it out and determine the position of the mean sun, we must add it. Thus, we find that $\angle E \Theta Z=79.29^{\circ}$.

Now that we know the true position of the sun on the eccentre at the time of the autumnal equinox in 2000, we can use the Table of Mean motion to figure out how much it would have advanced since the beginning of our epoch and subtract it from this position. The interval between noon on May 1, 1966 (the beginning of the epoch we are defining) and 11:30am September 22, 2000 is 34 years, 152 days ( 143 days in that calendar year +9 leap days), 23.5 hours. To determine how far the sun moved on the eccentre, we look up corresponding time periods in the Table of Mean Motion of the Sun in Appendix B, breaking up periods into ones present in the table and estimating between entries where necessary as shown in Table 2.1.

| Interval | Degrees |
| :--- | :--- |
| 18 years | 355.6237794 |
| 16 years | 356.1100261025 |
| 150 days | 147.8452917663 |
| 2 days | 1.9712705569 |
| 23 hours | 0.9445671418 |
| 0.5 hours | 0.0205340683 |
| Total | $\mathbf{8 6 2 . 5 1 5 4 6 9 0 3 5 8}$ |

Table 2.1: Motion of the sun on the eccentre between epoch and the 2000 autumnal equinox.

Note that the total has gone more than two full revolutions of $360^{\circ}$ so we will need to subtract out those revolutions to get that the relevant motion on the eccentre during that time is $\approx 142.52^{\circ}$. Subtracting that from the relative position of the sun in relation to the apogee on that date,

$$
79.29^{\circ}-142.52^{\circ}=-63.23^{\circ}
$$

This indicates that at the beginning of the epoch, the sun should have been $63.23^{\circ}$ before (i.e., clockwise from) the apogee on the eccentre which is the position of the mean sun.

We can translate that into an actual position of ecliptic longitude. Since we showed in Chapter 2.2 that the position of the sun at apogee had an ecliptic longitude of $102.59^{\circ}, 63.23^{\circ}$ before that would put the mean sun at $39.36^{\circ}$ ecliptic longitude. However, we must apply the adjustment for the anomaly by looking up the angle of the mean sun before epoch in the Table of Solar Anomaly (Appendix C). Since the mean sun is $63.23^{\circ}$ before the apogee, we must interpolate between the lines for 60 and 66 . Doing so, we get an equation of anomaly of $1.69^{\circ}$. Since our argument was found in the first column, we add that to the position of the mean sun:

$$
39.36^{\circ}+1.69^{\circ}=41.05^{\circ}
$$

Again, we can compare this to the actual values from Stellarium [7] (41.34 ) and the IMOs calculator [12] $\left(41.30^{\circ}\right)$ both indicating this model makes a prediction that is about $\frac{1}{3}{ }^{\circ}$ low. This is an excellent agreement and would not be easily detectable by instruments in Ptolemy's time.

### 2.5 Sample Calculation

Now that we have determined the position of the sun on the eccentre for the start of our newly defined epoch, we can explore how to use this information to calculate the position of the sun
on a given date. In short, it is the reverse of the process to calculate the position at epoch. We will first determine the interval of time since epoch, look up the mean motion in the Solar Mean Motion table (Appendix B), add that to position of the mean sun at epoch, determine how far that is before apogee, use that value to look up the equation of anomaly in the Table of Solar Anomaly (Appendix C), and add or subtract it from the mean position as necessary.

To begin, we will select the date of May 15, 2003 at $9: 45 \mathrm{pm} \mathrm{CST}^{12}$. We first determine the interval since the beginning of the epoch. This is a period of 37 years, 23 days ( 14 days in that calendar year +9 leap days), and 9.75 hours. These intervals then get entered into the Solar Mean Motion Table (Appendix B) and added:

| Interval | Degrees |
| :--- | :--- |
| 36 years | 351.2475587 |
| 1 year | 359.7568766314 |
| 23 days | 22.6696114042 |
| 9 hours | 0.3696132165 |
| 0.75 hours | 0.0308011121 |
| Total | $\mathbf{7 3 4 . 0 7 4 4 6 1 0 6 4 2}$ |

Table 2.2: Calculation of the increase in solar position on the eccentre due to mean motion between the beginning of the epoch and May 15, 2003 at 9:45pm.

Again, we subtract out any full revolutions of $360^{\circ}$ which leaves us with an increase of the mean solar position by $14.07^{\circ}$.

This gets added to the position of the mean at the start of the epoch which we showed in Chapter 2.4 to be $39.36^{\circ}$. Thus, the position of the mean sun on the date in question is $53.43^{\circ}$ ecliptic longitude.

Next, we determine the argument of anomaly which is how far past apogee the sun is. As we showed previously, the position of the apogee is $102.59^{\circ}$ ecliptic longitude. This means the sun has completed a revolution. In it, it moved $360^{\circ}-102.59^{\circ}=257.41^{\circ}$ and then moved another $53.43^{\circ}$ into its next revolution. Thus, $257.41^{\circ}+53.43^{\circ}=310.84^{\circ}$. This is the distance from apogee which we enter into the Table of the Sun's Anomaly (Appendix C), estimating between the values for rows 306 and 312 to determine an equation of anomaly of $1.43^{\circ}$. Since this value was found in the second column, we add it to the position of the mean sun on the date in question:

$$
53.43^{\circ}+1.43^{\circ}=54.86^{\circ}
$$

[^10]This puts the sun most of the way through Aries towards Taurus using the modern zodiac.
Checking this, we find a value of $54.80^{\circ}$ from Stellarium [7] and $54.85^{\circ}$ from the IMO's calculator [12], both of which are in excellent agreement with the derived value ${ }^{13}$.

### 2.6 Accuracy of the Model

Before closing out this chapter, we can do a final appraisal of how accurate Ptolemy's solar model truly is. To explore this, I created an Excel file that calculated the position of the sun at noon each day, beginning with the values derived for the start of the epoch and continuing through 2029. Some liberties were taken such that I did not use the Solar Mean Motion table or the Table of Solar Anomaly. Rather, the mean position was incremented by the daily motion discussed in Chapter 2.1 and the anomaly calculated each day using the method described in Chapter 2.3. Using Excel to calculate these more directly avoids the error caused by numerous steps of rounding along the way, giving a clearer picture of how the model truly behaves.

To determine the ecliptic longitude of the sun each day using a modern model, I turned to the Solar Position Algorithm at the National Renewable Energy Laboratory's (NREL) Measurement and Instrumentation Data Center (MIDC) [8] which was able to generate daily results for a wide range of years at one time.

Both of these datasets were imported into Microsoft Power BI, a data analysis tool, and graphed together. A sample, filtered to 2003, is shown in Figure 2.4.

In the plot of ecliptic longitude, the two lines are indistinguishable. Only in the delta can we see the difference where, in 2003, the models differed from $-0.02^{\circ}$ to $0.03^{\circ}$. As such, the minor variation between models found in Chapter 2.5 was largely a result of the rounding and estimations between values in the tables used.

[^11]

Figure 2.4: Graph of the ecliptic longitude of the sun on each day of the year for 2003 as based on the calculations in this paper (orange) and a modern astronomical model (blue). Below is a graph of the difference between the two.

If we do not filter the data to a single year, we can see the pattern longer term as shown in Figure 2.5 . Here, we can clearly see a yearly variation in the discrepancies which have a deviation of about $0.05^{\circ}$ in a single year. This is imprinted on a longer term pattern where the Ptolemaic model tends to get further ahead as time passes, having the most accurate year in 2000 which, as we noted before, was the year for which the model was calibrated.


Figure 2.5: Graph showing the variance between the Ptolemaic model and the modern one.

Ultimately, this shows the excellent agreement that the Ptolemaic model has with reality, accumulating $\approx 0.15^{\circ}$ of error every 30 years. Within a century, this would certainly be noticeable to astronomers in period as it would significantly change the timing and nature of eclipses predicted using this model.

## CHAPTER 3

## First Lunar Model

With the solar model complete, Ptolemy turns to the next brightest object in the sky: the moon. As with the solar model, Ptolemy will need to precisely know the position of the moon on given dates to calibrate the lunar model. Instrumentation of his time was unable to deliver the necessary precision, so Ptolemy turned to geometry. Specifically, it was understood that lunar eclipses only happened when the moon passed through the earth's shadow which is always directly opposite the sun. Thus, if an eclipse occurred, Ptolemy could determine the position of the moon at mid-eclipse as $180^{\circ}$ away from the sun, using the position of the sun calculated in the solar model, hence why the lunar model necessarily follows the solar one.


Figure 3.1: The lunar model as Hipparchus envisioned it. The moon's sphere and deferent are shown in blue. Its epicycle in red. The ecliptic, tilted by $5^{\circ}$ with respect to the moon's circle, is in yellow. The intersection of the two circles are the nodes with the ascending node nearer to us in this diagram.

In the Almagest, Ptolemy takes two books to develop his lunar model. In Book IV, he largely discusses the lunar model as it existed prior to him, developed by the Greek astronomer, Hipparchus, whose model is illustrated in Figure 3.1. Hipparchus made use of the epicyclic model, placing the earth at the center of the deferent and having the moon on an epicycle. In this model, the deferent rotates counter-clockwise while the epicycle rotates clockwise. In addition, this plane is tilted by $5^{\circ}$ with respect to the ecliptic. This small amount of tilt between these two circles is sufficiently small that Ptolemy chooses to ignore it as he develops this first model ${ }^{1}$.

In Book V, Ptolemy discusses discrepancies between the model just derived and observation, concluding that there is a second anomaly that must be introduced to fully account for the observations leading him to create a revised lunar model.

In this chapter, we will follow Ptolemy's methods beginning by exploring the lunar motions.

### 3.1 Lunar Motions

Before attempting to set up any model, Ptolemy explores the various periods of the moon the model will need to take into account. In other words, how do we define a lunar "month"? This becomes a complicated question as there are several ways to do so, all based on what Ptolemy referred to as "returns".

For example, we could ask how long it takes the moon to return to the same ecliptic longitude. This is known as a lunar sidereal month. Unlike the sun, whose motion is confined to the ecliptic ${ }^{2}$, the moon bobs up and down with respect to the ecliptic. As such, we could ask how long it takes the moon to return to the same ecliptic latitude from the same direction. For example, in one of these cycles, the moon will cross the ecliptic twice ( $0^{\circ}$ ecliptic latitude). However, once it will be moving upwards and once downwards. A full cycle would only include crossing the same ecliptic latitude from the same direction. This is known as a draconitic month. Much like the sun, the moon's speed appears to change as it orbits. The period it takes to return to the same speed is known as an anomalistic month. Lastly, we could consider the period it takes to return to the same phase. This is a lunation or a synodic month.

Unfortunately, the period of each of these different types of "month" are all different and none of them divide evenly into a solar year. This is why the moon phase is not the same on the same day of each year and the number of full moons in a year can vary.

Ptolemy explores these periods of lunar motion in Book IV, Chapter 2 (IV.2). In it, he turns to Babylonian astronomers who were excellent record keepers of astronomical phenomena. From these records, ancient astronomers attempted to find the cycles over which all of these various peri-

[^12]ods converged and then repeated. Then, by dividing this longer period by the number of individual returns, Ptolemy could determine with great accuracy the period of each cycle, which he expresses in degrees per interval of time in his table of lunar mean motions (IV.4). This table is reproduced in Appendix D.

### 3.2 Eclipse Details

To recreate Ptolemy's methods, we will need to select three eclipses. I have taken the following eclipses $^{3}$ from the NASA Javascript Lunar Eclipse Explorer[3], choosing eclipses that would have been visible in the Barony of Three Rivers.

1. May 15 2003, at $9: 45 \mathrm{pm}$
2. February 20 2008, at $9: 30 \mathrm{pm}$
3. April 15,2014 , at $1: 45 \mathrm{am}^{4}$

Following Ptolemy's method, we first calculate the solar position at these times which I have done using the results from the previous chapter on the solar epoch.

1. $54.86^{\circ}$ ecliptic longitude ${ }^{5}$
2. $331.99^{\circ}$ ecliptic longitude ${ }^{6}$
3. $25.21^{\circ}$ ecliptic longitude ${ }^{7}$

These can then be translated into the lunar position by adding or subtracting $180^{\circ}$ since the moon is opposite the sun.

1. $234.86^{\circ}$ ecliptic longitude
2. $151.99^{\circ}$ ecliptic longitude
3. $205.21^{\circ}$ ecliptic longitude
[^13]
### 3.3 Calibration of the Lunar Model

Now that we have the basic information for three eclipses selected, we can begin to work out the parameters of the lunar model. We currently have the true position of the moon at three times. However, these true positions are the combination of the mean and true position. Thus, to be able to understand the components of the model, we need to know where the mean moon was, as well as understand where the moon was on its epicycle, and the size of the epicycle and deferent.

### 3.3.1 Eclipse Pairing

To do so, Ptolemy pairs off the eclipses. If we take the first two eclipses from our selection we see that the moon advanced $277.13^{\circ}$ beyond full revolutions in the time between the two eclipses. That time span is 4 years, 281 days, and 23.75 hours. Similarly, we can pair the second and third eclipses to see it advanced $53.22^{\circ}$, beyond full revolutions, over the course of 6 years, 55 days days, and 4.25 hours.

These motions can then be compared to the mean motions over those intervals. To do so, we'll turn to the Lunar Mean Motion Table in Appendix D, looking at the increase in longitude as well as anomaly over this period. For the first pair of eclipses, we have the following:

| Interval | Mean Motion $\left({ }^{\circ}\right)$ | Anomalistic Motion $\left({ }^{\circ}\right)$ |
| :--- | :--- | :--- |
| 4 years | 157.5180341 | 354.8749757 |
| 270 days | 317.6231981 | 287.5453722 |
| 11 days | 144.9402044 | 143.7148115 |
| 23.75 hours | 13.03912824 | 12.92888929 |
| Total | $\mathbf{2 7 3 . 1 2 0 5 6 4 8}$ | $\mathbf{7 9 . 0 6 4 0 4 8 6 9}$ |

Table 3.1: Calculation of the increase in mean lunar longitude and anomalistic motion between May 15 2003, at 9:45pm and February 20 2008, at 9:30pm

The mean motion of Table 3.1, can then be compared to the true motion of $277.13^{\circ}$ calculated above to determine that the moon advanced $4.01^{\circ}$ more than it would have based on mean motion alone as a result of a $79.06^{\circ}$ motion about the epicycle. In other words, a $79.06^{\circ}$ motion about the epicycle resulted in the moon moving $4.01^{\circ}$ more than it would have based on mean motion alone. Thus, the effect of the anomaly was forwards in ecliptic latitude (i.e., counter-clockwise on the deferent).

Doing the same for the second pair of eclipses which are separated by 6 years, 55 days, and 4.25 hours:

| Interval | Mean Motion $\left({ }^{\circ}\right)$ | Anomalistic Motion $\left({ }^{\circ}\right)$ |
| :--- | :--- | :--- |
| 6 years | 56.27705118 | 175.3124635 |
| 30 days | 35.29146645 | 31.949485 |
| 25 days | 329.4095554 | 326.6245715 |
| 4.25 hours | 2.333317684 | 2.313590715 |
| Total | $\mathbf{6 3 . 3 1 1 3 9 0 7 1}$ | $\mathbf{1 7 6 . 2 0 0 1 1 0 7}$ |

Table 3.2: Calculation of the increase in mean lunar longitude and anomalistic motion between February 20 2008, at 9:30pm and April 15, 2014, at 1:45am

From Table 3.2, we can see that the moon's true motion ( $53.22^{\circ}$ ) over this interval is $10.09^{\circ}$ less than the mean motion as a result of $176.20^{\circ}$ about the epicycle. In other words, a motion of $176.20^{\circ}$ about the epicycle resulted in the moon moving $10.09^{\circ}$ less than it would have based on mean motion alone. Again, the effect of the anomaly was forwards (i.e., counter clockwise on the deferent).

### 3.3.2 Determining the Radius of the Deferent

Using this information, we can now determine the radius of the deferent. To do so, we will ignore the motion about the deferent, only considering the motion on the epicycle and the effect it has on the anomaly, to construct a diagram shown in Figure 3.2.


Figure 3.2: An initial schematic for solving the lunar model showing the relative positions for each of the three eclipses on the epicycle. While the angles about the epicycle are to scale, the size anomaly is not as the epicycle is not drawn to the correct scale in relation to the deferent.

Here, we have the moon's deferent, shown as a dashed line, centered on the earth at $E$. On the deferent, we have the epicycle, centered on $D$. We can then consider the position of the moon on
the epicycle at the time of the three eclipses given above, $A, B$, and $C$ respectively. In addition, we have created point $F$ which is where $\overline{E C}$ intersects the epicycle.

From our calculations above, we know that $\angle A D B$, the motion about the epicycle from the first eclipse to the second, is $79.06^{\circ}$ which produces an anomaly, $\angle A E B=4.01^{\circ}$ in the forward (counter-clockwise on the deferent) direction. Similarly, $\angle B D C$, the motion about the epicycle from the second eclipse to the third is $76.20^{\circ}$ and produces an anomaly, $\angle B E C$, of $10.09^{\circ}$, this time in the rearwards (clockwise on the deferent) direction.

Our first step in calibrating the model will be to determine the distance between the earth and the center of the epicycle, $\overline{E D}$ as shown in Figure 3.3. In this diagram we have also labeled the perigee on the epicycle, $K$, as well as the apogee, $J$.


Figure 3.3: Figure 3.2 redrawn to highlight the third eclipse and include the line between the earth and center of the deferent

To determine $\overline{E D}$, Ptolemy makes use of two theorems from Euclid's Elements [4]. Specifically, he refers to theorem III. 36 and II.6. Taken together, they give us the following equation ${ }^{8}$ :

$$
\overline{C E} \cdot \overline{F E}+\overline{D K}^{2}=\overline{D E}^{2}
$$

To solve this, Ptolemy takes the radius of the epicycle, $\overline{D K}$, to be $60^{p}$. Thus, we will need to solve for $\overline{C E}$ and $\overline{F E}$. This is done by creating numerous additional right triangles, as shown in Figure 3.4, which can then be solved.

[^14]

Figure 3.4: Eclipse diagram with additional right triangles drawn in.

In this figure, I have temporarily removed the angles from the center of the epicycle to each of the eclipses for clarity. We have connected $A$ to $B$ and $F$. Point $F$ was also connected to $B$. Next, $\overline{E A}$ was extended such that a line drawn from $F$ would meet it perpendicularly at $H$. A perpendicular is also extended from $F$ such that it meets $\overline{E B}$ at point $G$. Finally, a line is extended from $A$ such that it meets $\overline{F B}$ perpendicularly at $I$.

To solve this, Ptolemy first focuses on $\triangle F E G$. This is a right triangle with hypotenuse $\overline{F E}$. As described in Section 1.4.1, Ptolemy solves his triangle by imagining a circle around them and in that triangle, the hypotenuse $(\overline{F E})$ will be the diameter, having a length of $120^{p}$. We can also determine $\angle F E G=10.09^{\circ}$ as it is the anomaly between those two eclipses. Using trigonometry we can then determine $\overline{F G}_{F E G}=21.02^{p}$. Here, I have added a subscript to remind us for which context the size of these parts is true, since it is only true within the context of this imaginary circle about this triangle and we will shortly be creating several other imaginary circles, each with their own contexts of which we will need to keep track. We will then be able to convert pieces from the own contexts in which we find them into the context about $\triangle F E G$ if we have a line segment in common by which to do so, as we shall see shortly. While the line segment length will change depending on which context we're considering at a given time, then angles are always preserved between contexts.

For now, let us consider $\angle C D B$ (more easily visible in Figure 3.2). We know that this angle is 176.20 as that was the motion about the epicycle between the two eclipses. This is equal in measure to $\operatorname{arc} C B$. That arc subtends $\angle B F C$ which is on the perimeter of the epicycle and thus
has half the measure of the central angle/arc. Therefore, $\angle B F C=88.10^{\circ}$. We can then determine its supplement, $\angle B F E=91.90^{\circ}$.

We can then focus on $\triangle F B E$. In it, we have determined $\angle B F E=91.90^{\circ}$ and $\angle F E B=$ $10.09^{\circ}$. Since a triangle has $180^{\circ}$, this means the remaining angle, $\angle F B E=78.01^{\circ}$.

We can now turn our attention towards $\triangle F B G$ which contains the angle we just determined. In this context, $\overline{F B}_{F B G}=120^{p}$ as it is again the hypotenuse. Using the angle and the hypotenuse we can determine, using trigonometry, that $\overline{F G}_{F B G}=117.38^{p}$.

We will now convert the length of this line into the context we created about $\triangle F E G$. To do so, we will use the line segment we have determined in both contexts, $\overline{F G}$, to appropriately scale the unknown segment to the context of of $\triangle F E G$. This is done by taking the ratio of the the segment known in both contexts and setting it equal to the ratio of the unknown segment, taking care to keep the contexts (which I have denoted in subscripts) consistent as to whether they are in the numerator or denominator. In other words:

$$
\frac{\overline{F G}_{F E G}}{\overline{F G}_{F B G}}=\frac{\overline{F B}_{F E G}}{\overline{F B}_{F B G}}
$$

Substituting in:

$$
\frac{21.02^{p}}{117.38^{p}}=\frac{\overline{F B}_{F E G}}{120^{p}}
$$

Solving this, we find $\overline{F B}_{F E G}=21.49^{p}$. This procedure of solving triangles and converting them to the context of $\triangle F E G$ will be repeated several more times.

Next, we will perform it on $\triangle F E H$. There, the hypotenuse, $\overline{F E}_{F E H}=120^{p}$. We can also determine that $\angle A E F=6.08^{\circ}$ as it is the difference of the anomalies between each of the eclipse pairs: $\angle B E C-\angle A E B$. Using trigonometry, we find that $\overline{F H}_{F E H}=12.71^{p}$. In this case, we do not need to do any work to convert the contexts because our line segment in common, $\overline{F E}=120^{p}$ in both cases. Thus, the conversion ratio would be 1 . Thus, $\overline{F H}_{F E G}=12.71^{p}$.

Now, arc $A B C$ is the sum of $\angle A D B+\angle B D C$, the motion about the epicycle between the eclipses, and is therefore $225.26^{\circ}$. This arc subtends $\angle A F C$ which is on the perimeter of the circle and not on the arc itself. Thus, this angle has half the measure, which is to say, $\angle A F C=127.63^{\circ}$. We can then find its supplement along $\overline{C E}, \angle A F E$, to be $52.37^{\circ}$.

This gives us two of the angles in $\triangle F E A: \angle A F E=52.37^{\circ}$ and $\angle A E F=6.08^{\circ}$. Thus the remaining angle, $\angle F A E=121.55^{\circ}$. We can then take its supplement along $\overline{E H}$ and find that $\angle F A H=58.45^{\circ}$.

This allows us to focus on $\triangle F H A$. In that context, $\overline{F A}_{F H A}=120^{p}$. Again, using trigonometry, we can find that $\overline{F H}_{F H A}=102.26^{p}$. We can then convert $\overline{F A}_{F H A}$ into the context of $\triangle F E G$ using the following ratio with $\overline{F H}$ as our common segment.

$$
\frac{12.71^{p}}{102.26^{p}}=\frac{\overline{F A}_{F E G}}{120^{p}}
$$

Solving, we determine $\overline{F A}_{F E G}=14.91^{p}$.
Now consider arc $A B$ which has a measure of $79.06^{\circ}$. The angle on the perimeter of the circle that this arc subtends, $\angle A F B$, then has half the measure and is therefore $39.53^{\circ}$. This is part of $\triangle F A I$. We can now build a new context around this triangle in which $\overline{F A}_{F A I}=120^{p}$. From this, we can determine $\overline{A I}_{F A I}=76.38^{p}$ and $\overline{I F}_{F B I}=92.55^{p}$ using trigonometry. These two segments can then be converted back to the context of $\triangle F E G$ using $\overline{F B}$ as the common segment. Doing so we find that $\overline{A I}_{F E G}=9.49^{p}$ and $\overline{I F}_{F E G}=11.50^{p}$.

We can now determine a few other segments without having to solve additional triangles and change contexts. First, $\overline{B I}_{F E G}=\overline{F B}_{F E G}-\overline{I F}_{F E G}=9.99^{p}$. In addition, we can determine $\overline{A B}$ as it is part of right triangle $\triangle A I B$ in which we now know two sides. Thus, we can use the Pythagorean theorem to determine $\overline{A B}_{F E G}=13.78^{p}$.

Now that we have collected many of these segments in the context of $\triangle F E G$ we can convert them all to the context of the epicycle. In the epicycle, which has a radius of $60^{p}$, we can determine the length of $\overline{A B^{9}}$ since we know that $\angle A D B=\operatorname{arc} A B=79.06^{\circ}$. Either trigonometrically or using Ptolemy's table of chords, we can therefore look up the corresponding chord, $\overline{A B}=76.38^{p}$. This can now be used as our common segment to convert $\overline{F E}_{F E G}$.

$$
\frac{\overline{76.38^{p}}}{13.78^{p}}=\frac{\overline{F E}}{120^{p}}
$$

Solving this equation we find that $\overline{F E}=665.14^{p}$ when the radius of the epicycle is $60^{p}$.
Next, we will determine $\overline{C F}$ so that it can be added to $\overline{F E}$ to determine $\overline{C E}$. To do so, we will first need to convert $\overline{A F}$ to the context of the epicycle:

$$
\frac{\overline{76.38^{p}}}{13.78^{p}}=\frac{\overline{A F}}{14.91^{p}}
$$

Solving, we find $\overline{A F}=82.64^{p}$. From this chord, we can then determine $\angle A D F=84.05^{\circ}$, either using trigonometry or using Ptolemy's table of chords and interpolating between rows. This is the same measure as arc $A F$.

We can now determine $\operatorname{arc} C F$ as it is $360^{\circ}-\operatorname{arc} A F-\operatorname{arc} A B-\operatorname{arc} B C=50.69^{\circ}$. We can then determine the corresponding chord, $\overline{C F}$, again either trigonometrically or using Ptolemy's table of chords, to be $51.39^{p}$. This can then be added to $\overline{F E}$ to determine $\overline{C E}=716.51^{p}$.

We can now return to and solve the equation to determine the radius of the deferent, $\overline{D E}$ :

[^15]$$
716.51^{p 2} \cdot 665.14^{p 2}+60^{p^{2}}=\overline{D E}^{2}
$$

Solving this, we find $\overline{D E}=692.95^{p}$.
We can pause briefly to compare this value to Ptolemy's who derived the value twice. The first time, he used three eclipses recorded in Babylonian times, and the second set were ones from his own lifetime. This decision was made to ensure that the radius of the deferent was not changing over time. For the Babylonian eclipses, he found a value of $690.15^{p}$. For the second he found a value of $689.10^{p}$. Thus, the value derived here is in excellent agreement. As a reminder, these "parts" in which we have found this distance are still not in any absolute units but only for the context in which the radius of the epicycle is $60^{p}$.

### 3.3.3 Equation of Anomaly

Now that the distance between the earth and the center of the epicycle are determined, the anomaly in terms of the angle from apogee on the epicycle can easily be determined using trigonometry.


Figure 3.5: Diagram to find the equation of anomaly.

In Figure 3.5, we have again have the earth at $E$ and the center of the epicycle at $D$ with its apogee at $J$. If the moon is at some position, $L$, then the angle around the epicycle (remembering that the moon moves clockwise on the epicycle and measured from apogee) is $\angle J D L$ and $\overline{D L}=$ $60^{p}$. As we calculated in Section 3.3.2, $\overline{D E}=692.95^{p}$.

We can calculate $\overline{L M}$ as $60 \cdot \sin (\angle J D L)$. Then, the equation of anomaly can be calculated as $\tan ^{-1}\left(\frac{\overline{L M}}{692.95^{p}}\right)$.

The results of this calculation have been done, as for the sun, and are collected in Appendix E.

### 3.3.4 Epoch Position

Next, we will determine the epoch position, but will first need to deconstruct the true position for one of our three eclipses into the components of the mean position and the anomaly. To do so, we will make use of Figure 3.6.


Figure 3.6: Diagram to find the anomaly for the third eclipse.

In this diagram, we have removed the first two eclipses. We have extended a line from the center of the epicycle, $D$, such that it meets $\overline{C E}$ perpendicularly at $N$, which will necessarily bisect it as a consequence of the perpendicular bisector theorem [6]. Thus, $\overline{N F}=\frac{1}{2} \overline{C F}$. Since we determined $\overline{C F}=51.39^{p}$ in Section 3.3.2, this indicates $\overline{N F}=25.70^{p}$. This can then be added to $\overline{F E}$ which we previously determined to be $665.14^{p}$ to determine $\overline{N E}=690.84^{p}$.

In addition, we previously determined that $\overline{E D}=692.95^{p}$. In, $\triangle N E D$, this is the hypotenuse, so we can now use trigonometry to determine $\angle N E D$ :

$$
\angle N E D=\cos ^{-1}\left(\frac{690.84^{p}}{692.95^{p}}\right)=4.47^{\circ}
$$

Recall that, in Section 3.2, we calculated the true position of the moon in ecliptic longitude for the third eclipse was $205.21^{\circ}$ ecliptic longitude. Since this anomaly is clearly subtractive based on
the diagram, we add it to the true position to determine the mean position to be $209.68^{\circ}$ ecliptic longitude.

In addition, we can determine the position of the moon about the epicycle from Appendix E by looking up the anomaly and interpolating between rows to determine the angle from apogee, recalling that the anomaly was subtractive and as such, we must look in the first column, closer to apogee than perigee. Doing so, we find the angle from apogee to be $64.65^{\circ}$ at the time of the eclipse.

With these two values in hand, we can now determine the epoch position as we did with the sun. First, we need to determine the interval of time between the beginning of the epoch and the time of the eclipse. That duration is 47 years, 360 days, and 13.75 hours. We can then use our lunar table of mean motion (Appendix D) to determine the motions over this time period.

| Interval | Mean Motion $\left({ }^{\circ}\right)$ | Anomalistic Motion $\left({ }^{\circ}\right)$ |
| :--- | :--- | :--- |
| 36 years | 337.6623071 | 313.8747812 |
| 11 years | 343.1745938 | 255.9061831 |
| 360 days | 63.49759745 | 351.4443438 |
| 13.75 hours | 7.548968978 | 23.39382962 |
| Total | $\mathbf{3 1 . 8 8 3 4 6 7 3 3}$ | $\mathbf{2 2 4 . 6 1 9 1 3 7 7}$ |

Table 3.3: Calculation of the increase in mean lunar longitude and anomalistic motion between May 1, 1966 at noon and, at 9:30pm and April 15, 2014, at 1:45am.

This increase in mean motion can then be subtracted from the mean position to determine the position of the mean moon at the beginning of the epoch. Doing so, we find that the mean moon was at $177.80^{\circ}$ in ecliptic longitude. In addition, we can determine the position of the moon on the epicycle, measured clockwise from apogee, to be $199.73^{\circ}$.

### 3.4 Sample Calculation

As a sample calculation, we will determine the position of the moon in ecliptic longitude at 5:30am on February 14, 2001 ${ }^{10}$. To begin, we determine the amount of time between the beginning of the epoch and this time to be 34 years, 297 days, and 18.5 hours ${ }^{11}$.

[^16]| Interval | Mean Motion $\left({ }^{\circ}\right)$ | Anomalistic Motion $\left({ }^{\circ}\right)$ |
| :--- | :--- | :--- |
| 18 years | 168.8311535 | 156.9373906 |
| 16 years | 270.0671365 | 339.499625 |
| 270 days | 317.6231981 | 287.5453722 |
| 27 days | 355.7623198 | 352.7545372 |
| 18 hours | 9.333270736 | 9.254362859 |
| 0.5 hours | 0.274507963 | 0.272187143 |
| Total | $\mathbf{4 1 . 8 9 1 5 8 6 6}$ | $\mathbf{6 6 . 2 6 3 4 7 5}$ |

Table 3.4: Calculation of the increase in mean lunar longitude and anomalistic motion between May 1, 1966 at noon and, at 5:30am on February 14, 2001.

From this, we can see that the moon advanced $41.89^{\circ}$ beyond a full revolution on the deferent and $66.26^{\circ}$ beyond full revolutions about the epicycle.

To each of these, we will add their positions at epoch.

| Interval | Mean Motion $\left({ }^{\circ}\right)$ | Anomalistic Motion $\left({ }^{\circ}\right)$ |
| :--- | :--- | :--- |
| Epoch Position | 177.8 | 199.73 |
| Interval motion | 41.8915866 | 66.263475 |
| Total | $\mathbf{2 2 0 . 2 4 0 6 0 2 5}$ | $\mathbf{2 6 6 . 5 3 7 8 4 9 3}$ |

We then use the value of the anomalistic to determine the anomaly by looking this value up in the Table of Lunar Anomaly (Appendix E). This produces an anomaly of $4.939^{\circ}$ and since the value we looked up was in the second column, it is additive for a final ecliptic longitude of $225.180^{\circ}$ from this model.

### 3.5 Accuracy of the Model

To investigate the accuracy of the lunar model, I created an Excel file which incremented the motion on the deferent as well as that of the epicycle, then used the position on the epicycle to calculate and apply the anomaly to determine the ecliptic longitude of the moon each day at noon. This was then compared to a data set from NASA’s Horizon app [9] also giving the ecliptic longitude for the moon each day at noon. The variance between them for 2020-2021 is shown in Figure 3.7.


Figure 3.7: Plot of the variance between the position of the moon as predicted by the first lunar model and position as calculated using NASA Horizon app.

Very quickly from this figure, we can see that the first lunar model has significant problems as it can disagree with the true position by as much as $-3.9^{\circ 12}$ in the years between the years 1966 and 2029. However, there is something interesting we can notice if we take into consideration the phase of the moon on the above graph.


Figure 3.8: Figure 3.7 with addition of full moons (open circles) and new moons (filled circles).

Here, we can see that the fit of the model is actually quite reasonable when the moon is full (at opposition) or new (at conjunction). In fact, the agreement is even better than suggested by this image alone because the points plotted are for noon on the day of the phase and can therefore differ from the actual moment of full or new phase by as much as 12 hours and since the slope of the variance is greatest around these times, this offset in time will play a significant impact. This agreement at conjunction and opposition (together known as syzygy) is unsurprising since the model was calibrated using lunar eclipses which happen only at opposition.

Between the full and new phases are the first and third quarter moons which is where the model is deviating most strongly. Ptolemy was aware of this discrepancy, and addressing it is the subject of Book V of the Almagest which we shall address in the next chapter.

[^17]
## APPENDIX A

## Glossary of Terms

## Altitude Circle

A great circle through an observer's zenith, an object, and the horizon. The angle above the horizon is measured along this circle.

## Anomaly/Anomalistic Motion

If the sun's position could be tracked against the background stars daily, it would be found that some days, it moves more in ecliptic longitude than others. The average speed at which it travels is the mean motion and the variance either above or below this is the anomalistic motion. This difference in motion produces an angular difference in the position of the sun compared to where the sun would have been due to mean motion alone. This angular difference is generally referred to as the anomaly or equation of anomaly.

## Apogee

In both of Ptolemy's models, there are times at which objects paths along their spheres will take them furthest from Earth. The furthest point is known at apogee.

## Argument

An argument is a value, often looked up in a table, to find another value. For example, an argument of anomaly would be looked up in one of the tables of anomaly to determine the equation of anomaly.

## Celestial Sphere

When observing the night sky, the apparent shape of the heavens is that of a dome above us. It was apparent to ancient astronomers that the Earth was a sphere, and thus the night sky could not simply be a dome above a flat surface, but had to be a sphere concentric to our own, albeit much larger. To this sphere, all stars were fixed, unmovable. Since these astronomers argued against the Earth turning on its axis to explain the daily motion of the night sky, they instead required that the stellar sphere rotated once every day.
Celestial Equator
As the celestial sphere rotates, it appears to have fixed poles around which it does so, just as the Earth does. Similarly, the celestial sphere also has a celestial equator, defined as a great circle $90^{\circ}$
away from these poles. Since the poles of the celestial sphere's rotation are directly above Earth's so too is the celestial equator the direct projection of Earth's equator onto the celestial sphere.

## Celestial Poles (North/South)

The points on the celestial sphere directly above the earth's north and south poles about which the celestial sphere seems to rotate.

## Circles

Following Ptolemy, this paper frequently makes reference to circles. However, Ptolemy's vision of the heavens truly involved crystalline spheres. But since objects were fixed to the surface of these spheres, the rotation of a sphere would produce motion in a single plane which would define a circle.

As did astronomers before him, Ptolemy divides the circle into $360^{\circ}$. Arcs along the circumference are similarly measured in degrees equal to the central angle which the arc subtends. However, lines within the circle are measured in an arbitrary unit of "parts". Regardless of the actual size of the circle, it was defined to have a radius of 60 parts. Because Ptolemy worked in a base 60 system known as sexagesimal, this is essentially the same as a circle having a radius of 1 , analogous to the modern unit circle.

## Conjunction

When two celestial objects have the same ecliptic longitude.

## Deferent

In the epicyclic model, the deferent is the main circle that carries the epicycles on which objects lie.

## Double Elongation

Twice the elongation. In Ptolemy's second lunar model, this is always the angular distance between the mean moon and the apogee of the deferent.

## Eccentre

In Ptolemy's eccentric model, the eccentric is the circle on which the celestial object travels. This circle has its center offset from the Earth. The circle rotates counter clockwise.

## Ecliptic

The apparent path of the sun in relation to the background stars. This forms a great circle and the constellations through which the sun travels are the zodiac constellations.

## Ecliptic Coordinates (Latitude/Longitude)

To describe the position of objects in the sky requires a coordinate system by which things can be measured. This coordinate system should be affixed to the sky, so stars at least retain a fixed position. One of the more commonly used ones in period was the ecliptic position in which a star's position was measured in relation to the ecliptic. If it was to the north of the ecliptic, it would have a positive ecliptic latitude. If to the south, it would be negative.

The system also needed a left-right component. But to do so required a zero point. This was chosen as the point at which the ecliptic and celestial equator cross when the sun is in spring (i.e., the vernal equinox). If we placed ourselves as observers outside of the celestial sphere and looked down on the ecliptic as a flat circle, we would measure the angle from this point counter-clockwise. From within the celestial sphere, this would be described as right to left. This convention is chosen because the mean motion of the sun and other celestial objects is right to left in relation to the background stars. This is often described as "rearwards" in the Almagest. Thus, if one were to say that an object were "to the rear of" or "after" a point, this would mean "counter clockwise from" if viewed from outside the celestial sphere, or "to the left of" from within. Conversely, if an object were "in advance of" or "before" a point, this would mean "clockwise from" or "to the right of."

Since the ecliptic is defined by the sun's motion, the sun will always be at $0^{\circ}$ ecliptic latitude, but is constantly changing its longitude.

## Egyptian Calendar

The Egyptian calendar was divided into 12 months, each consisting of 30 days. This gives a total of 360 days in the main year, but the Egyptian year has 365 days. This results in 5 "epagomenal" days being inserted at the end of each year which are not defined as being part of any month. This calendar did not include leap years.

## Elongation

The difference in ecliptic latitude between two celestial objects (typically the object and the sun).

## Epicycle

In order to explain the anomalistic motion of objects, ancient astronomers placed a secondary circle on the deferent. This circle would rotate, sometimes putting the apparent position ahead of the mean motion, and behind at other times. The size, rotation direction, and rotation speed could be fine tuned to help explain the objects motions.

Equation of Anomaly The amount that the position of a celestial object differs from the mean motion due to one of the various methods Ptolemy employs to induce anomalies in his models.

## Equinox

The name equinox translates to "equal night" which indicates that the night and day are of equal duration on these special dates. This occurs because the sun's position on the ecliptic is at one of the two nodes at which it intersects the celestial equator. Thus, if you were on Earth's equator, the sun would be directly overhead at local noon.
Great Circle A circle on the celestial sphere which has a center coincident with the center of the celestial sphere and therefore the same diameter.

## Mean Apogee

The point on the epicycle furthest from the point on the circle carrying the center of the deferent
diametrically opposite the center of the deferent from the earth.

## Mean Motion/Position

Annually, the sun completes a full circuit of its path along the ecliptic. Since this path is defined as $360^{\circ}$ and this happens in a little over 365 days, the daily motion averages to be just under $1 \%$ day. This is the mean motion. The position of the sun based on this average alone (ignoring the anomalistic motion) is the mean position.

## Meridian

A great circle across the celestial sphere that intersects the observers horizon due north and south (and thus crossing through the north and south celestial poles) and crossing through their zenith.

## Opposition

When two celestial objects have ecliptic longitudes that are $180^{\circ}$ apart.

## Parallax

The shift in apparent position of an object due to the observer's position. Most frequently in the Almagest, this is considered for the moon as it has a significant parallax due to the observer being on the surface of the Earth which is not at the center of the celestial sphere.

## Perigee

In both the epicyclic and eccentric models of motion, the object is brought closer and further from the Earth. The closest point is known as perigee.

## Quadrature

When an object has an elongation of $90^{\circ}$ from another (typically from the sun).

## Retrograde Motion

In general, the sun, moon, and planets all move right-to-left amongst the background stars when viewed from earth (counter-clockwise when viewed from above the celestial sphere). However, planets can sometimes slow and reverse their direction moving left-to-right. This apparent change in direction is known as retrograde motion.

## Small Circle

Any circle drawn on the celestial sphere which does not have its center as the center of the celestial sphere and therefore has a smaller diameter.

## Solstice

Since the ecliptic is tilted with respect to the celestial equator, there are two points when it has its extremes in distance away from the celestial equator. One when the sun is the most northwards and another when it is the most southwards. These points define the solstices. When viewed from Earth, this results in the sun being higher in summer and lower in winter (in the northern hemisphere). In addition, it results in the sun tracing a larger arc above the horizon in summer and shorter in winter which results in the summer solstice being the longest day and the winter solstice
being the shortest.

## Syzygy

A conjunction or opposition of two celestial bodies, usually with the sun as one of the objects.

## True Apogee

The point on the epicycle furthest from an observer on earth.

## Zenith

The point in the sky for an observer $90^{\circ}$ from their horizon (i.e., straight up).

## Zodiac

The ecliptic traces a path through 12 constellations. These constellations are the 12 familiar zodiac signs. Despite the constellations being notably different sizes in the sky, each is defined as taking up $30^{\circ}$ of the ecliptic. In Ptolemy's time, the projection of the sun's position onto the ecliptic was just entering Aries on the vernal equinox, but due to precession of the equinoxes, it now resides near the beginning of Pisces at this time of year.

## APPENDIX B

## Table of the Mean Motion of the Sun

This appendix reproduces table III. 2 of the Almagest which gives the mean motion of the sun in degrees for various periods. This table is not updated from that of Ptolemy as it is based on the length of a tropical year. Since the goal of this project is to understand how Ptolemy's models would have worked, this value was not updated in turn implying no need to update this table. Here, it is presented in decimal form instead of sexagesimal for ease of use and arranged in decreasing intervals.

| 18-year periods | Degrees |
| :--- | :--- |
| 18 | 355.6237794 |
| 36 | 351.2475587 |
| 54 | 346.8713381 |
| 82 | 342.4951175 |
| 90 | 338.1188968 |
| 108 | 333.7426762 |
| 126 | 329.3664556 |
| 144 | 324.9902349 |
| 162 | 320.6140143 |
| 180 | 316.2377937 |
| 198 | 311.861573 |
| 216 | 307.4853524 |
| 234 | 303.1091317 |
| 252 | 298.7329111 |
| 270 | 294.3566905 |
| 288 | 289.9804698 |
| 306 | 285.6042492 |
| 324 | 281.2280286 |


| 342 | 276.8518079 |
| :---: | :---: |
| 360 | 272.4755873 |
| 378 | 268.0993667 |
| 396 | 263.723146 |
| 414 | 259.3469254 |
| 432 | 254.9707048 |
| 450 | 250.5944841 |
| 468 | 246.2182635 |
| 486 | 241.8420429 |
| 504 | 237.4658222 |
| 522 | 233.0896016 |
| 540 | 228.713381 |
| 558 | 224.3371603 |
| 576 | 219.9609397 |
| 594 | 215.5847191 |
| 612 | 211.2084984 |
| 630 | 206.8322778 |
| 648 | 202.4560572 |
| 666 | 198.0798365 |
| 684 | 193.7036159 |
| 702 | 189.3273952 |
| 720 | 184.9511746 |
| 738 | 180.574954 |
| 756 | 176.1987333 |
| 774 | 171.8225127 |
| 792 | 167.4462921 |
| 810 | 163.0700714 |
| Single Years | Degrees |
| 1 | 359.7568766314 |
| 2 | 359.5137532628 |
| 3 | 359.2706298942 |
| 4 | 359.0275065256 |
| 5 | 358.784383157 |
| 6 | 358.5412597885 |


| 7 | 358.2981364199 |
| :---: | :---: |
| 8 | 358.0550130513 |
| 9 | 357.8118896827 |
| 10 | 357.5687663141 |
| 11 | 357.3264762788 |
| 12 | 357.0825195769 |
| 13 | 356.8393962083 |
| 14 | 356.5962728397 |
| 15 | 356.3531494711 |
| 16 | 356.1100261025 |
| 17 | 355.8669027339 |
| 18 | 355.6237793654 |
| Days (Months) | Degrees |
| 30 | 29.5690583533 |
| 60 | 59.1381167065 |
| 90 | 88.7071750598 |
| 120 | 118.2762334131 |
| 150 | 147.8452917663 |
| 180 | 177.4143501196 |
| 210 | 206.9834084729 |
| 240 | 236.5524668261 |
| 270 | 266.1215251794 |
| 300 | 295.6905835327 |
| 330 | 325.2596418859 |
| 360 | 354.8287002392 |
| Days | Degrees |
| 1 | 0.9856352784 |
| 2 | 1.9712705569 |
| 3 | 2.9569058353 |
| 4 | 3.9425411138 |
| 5 | 4.9281763922 |
| 6 | 5.9138116707 |
| 7 | 6.8994469491 |
| 8 | 7.885082459 |


| 9 | 8.870717506 |
| :---: | :---: |
| 10 | 9.8563527844 |
| 11 | 10.8419880629 |
| 12 | 11.826790008 |
| 13 | 12.8132586197 |
| 14 | 13.7988938982 |
| 15 | 14.7845291766 |
| 16 | 15.7701644551 |
| 17 | 16.7557997335 |
| 18 | 17.741435012 |
| 19 | 18.7270702904 |
| 20 | 19.7127055688 |
| 21 | 20.6983408473 |
| 22 | 21.6839761257 |
| 23 | 22.6696114042 |
| 24 | 23.6552466826 |
| 25 | 24.6408819611 |
| 26 | 25.6265172395 |
| 27 | 26.6121525179 |
| 28 | 27.5977877964 |
| 29 | 28.5834230748 |
| 30 | 29.5690583533 |
| Hours | Degrees |
| 1 | 0.0410681366 |
| 2 | 0.0821362732 |
| 3 | 0.1232044098 |
| 4 | 0.1642725464 |
| 5 | 0.205340683 |
| 6 | 0.2464088196 |
| 7 | 0.2874769562 |
| 8 | 0.3285450799 |
| 9 | 0.3696132165 |
| 10 | 0.410681366 |
| 11 | 0.4517495026 |


| 12 | 0.4928176392 |
| :--- | :--- |
| 13 | 0.5338857758 |
| 14 | 0.5749539124 |
| 15 | 0.616022049 |
| 16 | 0.6570901856 |
| 17 | 0.6981583222 |
| 18 | 0.7392264588 |
| 19 | 0.7802945954 |
| 20 | 0.821362732 |
| 21 | 0.8624308686 |
| 22 | 0.9034990052 |
| 23 | 0.9445671418 |
| 24 | 0.9856352784 |

## APPENDIX C

## Table of the Sun's Anomaly

In this table, I give the relationship between the angle of the mean sun past apogee to find the equation of anomaly. Here, "past apogee" means the number of degrees the sun is on the eccentre measured beginning at apogee and going counter-clockwise.

This table does not match the one Ptolemy calculated in III. 6 of the Almagest as the distance between the earth and the center of the sun's eccentre changed since Ptolemy's time. As such, these have been recalculated as described in Chapter 2.3 of this paper, using Excel.

To know whether the equation of anomaly should be additive or subtractive, determine which column the argument is in. If the argument is found in the first column, then the sun appears to lag where it should due to mean motion alone and the equation of anomaly should be subtracted. If it is found in the second column, then the sun would appear ahead of the position it should have due to the mean motion. As a result, the equation should be added to the mean motion.

| Angle from Apogee |  | Equation of Anomaly $\left({ }^{\circ}\right)$ |
| :--- | :--- | :--- |
| $\mathbf{6}$ | $\mathbf{3 5 4}$ | 0.195 |
| $\mathbf{1 2}$ | $\mathbf{3 4 8}$ | 0.388 |
| $\mathbf{1 8}$ | $\mathbf{3 4 2}$ | 0.578 |
| $\mathbf{2 4}$ | $\mathbf{3 3 6}$ | 0.761 |
| $\mathbf{3 0}$ | $\mathbf{3 3 0}$ | 0.937 |
| $\mathbf{3 6}$ | $\mathbf{3 2 4}$ | 1.104 |
| $\mathbf{4 2}$ | $\mathbf{3 1 8}$ | 1.259 |
| $\mathbf{4 8}$ | $\mathbf{3 1 2}$ | 1.402 |
| $\mathbf{5 4}$ | $\mathbf{3 0 6}$ | 1.530 |
| $\mathbf{6 0}$ | $\mathbf{3 0 0}$ | 1.642 |
| $\mathbf{6 6}$ | $\mathbf{2 9 4}$ | 1.738 |
| $\mathbf{7 2}$ | $\mathbf{2 8 8}$ | 1.815 |
| $\mathbf{7 8}$ | $\mathbf{2 8 2}$ | 1.873 |


| $\mathbf{8 4}$ | $\mathbf{2 7 6}$ | 1.911 |
| :--- | :--- | :--- |
| $\mathbf{9 0}$ | $\mathbf{2 7 0}$ | 1.928 |
| $\mathbf{9 3}$ | $\mathbf{2 6 7}$ | 1.929 |
| $\mathbf{9 6}$ | $\mathbf{2 6 4}$ | 1.924 |
| $\mathbf{9 9}$ | $\mathbf{2 6 1}$ | 1.915 |
| $\mathbf{1 0 2}$ | $\mathbf{2 5 8}$ | 1.899 |
| $\mathbf{1 0 5}$ | $\mathbf{2 5 5}$ | 1.879 |
| $\mathbf{1 0 8}$ | $\mathbf{2 5 2}$ | 1.853 |
| $\mathbf{1 1 1}$ | $\mathbf{2 4 9}$ | 1.822 |
| $\mathbf{1 1 4}$ | $\mathbf{2 4 6}$ | 1.786 |
| $\mathbf{1 1 7}$ | $\mathbf{2 4 3}$ | 1.745 |
| $\mathbf{1 2 0}$ | $\mathbf{2 4 0}$ | 1.699 |
| $\mathbf{1 2 3}$ | $\mathbf{2 3 7}$ | 1.648 |
| $\mathbf{1 2 6}$ | $\mathbf{2 3 4}$ | 1.592 |
| $\mathbf{1 2 9}$ | $\mathbf{2 3 1}$ | 1.531 |
| $\mathbf{1 3 2}$ | $\mathbf{2 2 8}$ | 1.466 |
| $\mathbf{1 3 5}$ | $\mathbf{2 2 5}$ | 1.397 |
| $\mathbf{1 3 8}$ | $\mathbf{2 2 2}$ | 1.324 |
| $\mathbf{1 4 1}$ | $\mathbf{2 1 9}$ | 1.246 |
| $\mathbf{1 4 4}$ | $\mathbf{2 1 6}$ | 1.165 |
| $\mathbf{1 4 7}$ | $\mathbf{2 1 3}$ | 1.081 |
| $\mathbf{1 5 0}$ | $\mathbf{2 1 0}$ | 0.993 |
| $\mathbf{1 5 3}$ | $\mathbf{2 0 7}$ | 0.903 |
| $\mathbf{1 5 6}$ | $\mathbf{2 0 4}$ | 0.809 |
| $\mathbf{1 5 9}$ | $\mathbf{2 0 1}$ | 0.714 |
| $\mathbf{1 6 2}$ | $\mathbf{1 9 8}$ | 0.616 |
| $\mathbf{1 6 5}$ | $\mathbf{1 9 5}$ | 0.516 |
| $\mathbf{1 6 8}$ | $\mathbf{1 9 2}$ | 0.415 |
| $\mathbf{1 7 1}$ | $\mathbf{1 8 9}$ | 0.312 |
| $\mathbf{1 7 4}$ | $\mathbf{1 8 6}$ | 0.209 |
| $\mathbf{1 7 7}$ | $\mathbf{1 8 3}$ | 0.104 |
| $\mathbf{1 8 0}$ | $\mathbf{1 8 0}$ | 0.000 |
|  |  |  |

## APPENDIX D

## Table of the Mean Motion of the Moon

This appendix is a copy of the Lunar Mean Motions table presented in Book IV Chapter 4. It is presented here in decimal as opposed to the original sexigesimal. In addition, it is presented in order of descending units of time whereas the original was presented out of order to help preserve space.

|  | Longitude | Anomaly | Latitude | Elongation |
| :--- | :--- | :--- | :--- | :--- |
| Epoch Position | $177.80^{\circ}$ | $199.73^{\circ}$ | ${ }^{\circ}$ | $\circ$ |
| $\mathbf{1 8}$ Year Periods | Longitude | Anomaly | Latitude | Elongation |
| $\mathbf{1 8}$ | 168.8311535 | 156.9373906 | 156.8360617 | 173.2073742 |
| $\mathbf{3 6}$ | 337.6623071 | 313.8747812 | 313.6721234 | 346.4147483 |
| $\mathbf{5 4}$ | 146.4934606 | 110.8121718 | 110.5081851 | 159.6221225 |
| $\mathbf{7 2}$ | 315.3246141 | 267.7495624 | 267.3442468 | 332.8294967 |
| $\mathbf{9 0}$ | 124.1557677 | 64.686953 | 64.18030846 | 146.0368708 |
| $\mathbf{1 0 8}$ | 292.9869212 | 221.6243436 | 221.0163702 | 319.244245 |
| $\mathbf{1 2 6}$ | 101.8180747 | 18.5617342 | 17.85243184 | 132.4516192 |
| $\mathbf{1 4 4}$ | 270.6492282 | 175.4991248 | 174.6884935 | 305.6589933 |
| $\mathbf{1 6 2}$ | 79.48038177 | 332.4365154 | 331.5245552 | 118.8663675 |
| $\mathbf{1 8 0}$ | 248.3115353 | 129.373906 | 128.3606169 | 292.0737416 |
| $\mathbf{1 9 8}$ | 57.14268883 | 286.3112966 | 285.1966786 | 105.2811158 |
| $\mathbf{2 1 6}$ | 225.9738424 | 83.2486872 | 82.0327403 | 278.48849 |
| $\mathbf{2 3 4}$ | 34.80499589 | 240.1860778 | 238.868802 | 91.69586414 |
| $\mathbf{2 5 2}$ | 203.6361494 | 37.1234684 | 35.70486368 | 264.9032383 |
| $\mathbf{2 7 0}$ | 12.46730295 | 194.060859 | 192.5409254 | 78.11061247 |
| $\mathbf{2 8 8}$ | 181.2984565 | 350.9982496 | 349.3769871 | 251.3179866 |
| $\mathbf{3 0 6}$ | 351.2962767 | 147.9356402 | 146.2130488 | 64.5253608 |
| $\mathbf{3 2 4}$ | 158.9607635 | 304.8730308 | 303.0491105 | 237.732735 |


| $\mathbf{3 4 2}$ | 327.7919171 | 101.8104214 | 99.88517214 | 50.94010913 |
| :--- | :--- | :--- | :--- | :--- |
| $\mathbf{3 6 0}$ | 136.6230706 | 258.747812 | 256.7212338 | 224.1474833 |
| $\mathbf{3 7 8}$ | 305.4542241 | 55.6852026 | 53.55729553 | 37.35485746 |
| $\mathbf{3 9 6}$ | 114.2853777 | 212.6225932 | 210.3933572 | 210.5622316 |
| $\mathbf{4 1 4}$ | 283.1165312 | 9.559983798 | 7.22941891 | 23.76960579 |
| $\mathbf{4 3 2}$ | 91.94768472 | 166.4973744 | 164.0654806 | 196.97698 |
| $\mathbf{4 5 0}$ | 260.7788383 | 323.434765 | 320.9015423 | 10.18435412 |
| $\mathbf{4 6 8}$ | 69.60989919 | 120.3721556 | 117.737604 | 183.3917283 |
| $\mathbf{4 8 6}$ | 238.4411453 | 227.3095462 | 274.5736657 | 356.5991024 |
| $\mathbf{5 0 4}$ | 47.27229884 | 74.2469368 | 71.40972737 | 169.8064766 |
| $\mathbf{5 2 2}$ | 216.1034524 | 231.1843274 | 228.2457891 | 343.0138508 |
| $\mathbf{5 4 0}$ | 24.9346059 | 28.121718 | 25.08185075 | 156.2212249 |
| $\mathbf{5 5 8}$ | 193.7657594 | 185.0591086 | 181.9179124 | 329.4285991 |
| $\mathbf{5 7 6}$ | 2.596912963 | 341.9964992 | 338.7539741 | 142.6359733 |
| $\mathbf{5 9 4}$ | 171.4280665 | 138.9338898 | 135.5900358 | 315.8433474 |
| $\mathbf{6 1 2}$ | 340.25922 | 295.8712804 | 292.4260975 | 129.0507216 |
| $\mathbf{6 3 0}$ | 149.0903736 | 92.808671 | 89.26215921 | 302.2580958 |
| $\mathbf{6 4 8}$ | 317.9215271 | 249.7460616 | 246.0982209 | 115.4654699 |
| $\mathbf{6 6 6}$ | 126.7526806 | 46.6834522 | 42.93428259 | 288.6728441 |
| $\mathbf{6 8 4}$ | 295.5838341 | 203.6208428 | 199.7703443 | 101.8802183 |
| $\mathbf{7 0 2}$ | 104.4149877 | 0.558233397 | 356.606406 | 275.0875924 |
| $\mathbf{7 2 0}$ | 273.2461412 | 157.495624 | 153.4424677 | 88.29496659 |
| $\mathbf{7 3 8}$ | 82.07729473 | 314.4330146 | 310.2785294 | 261.5023408 |
| $\mathbf{7 5 6}$ | 250.9084483 | 111.3704052 | 107.1145911 | 74.70971492 |
| $\mathbf{7 7 4}$ | 59.73960179 | 268.3077958 | 263.9506527 | 247.9170891 |
| $\mathbf{7 9 2}$ | 228.5707553 | 65.2451864 | 60.78671444 | 61.12446325 |
| $\mathbf{8 1 0}$ | 37.40190885 | 222.182577 | 217.6227761 | 234.3318374 |
| Single Years | Longitude | Anomaly | Latitude | Elongation |
| $\mathbf{1}$ | 129.3795085 | 88.71874392 | 148.7131145 | 129.6226319 |
| $\mathbf{2}$ | 258.7590171 | 177.4374878 | 297.4262291 | 259.2452638 |
| $\mathbf{3}$ | 28.13852559 | 266.1562318 | 86.13934362 | 28.86789569 |
| $\mathbf{4}$ | 157.5180341 | 354.8749757 | 234.8524582 | 158.4905276 |
| $\mathbf{5}$ | 286.8975426 | 85.59372038 | 23.56557269 | 288.1131595 |
| $\mathbf{6}$ | 175.3124635 | 172.2786872 | 57.73579139 |  |
|  |  |  |  |  |
|  |  |  |  |  |


| $\mathbf{7}$ | 185.6565597 | 261.0312075 | 320.9918018 | 187.3584233 |
| :--- | :--- | :--- | :--- | :--- |
| $\mathbf{8}$ | 315.0360682 | 349.7499514 | 109.7049163 | 316.9810552 |
| $\mathbf{9}$ | 84.41557677 | 78.4686953 | 258.4180308 | 86.60368708 |
| $\mathbf{1 0}$ | 213.7950853 | 167.1874392 | 47.13114538 | 216.226319 |
| $\mathbf{1 1}$ | 343.1745938 | 255.9061831 | 195.8442599 | 345.8489509 |
| $\mathbf{1 2}$ | 112.5541024 | 344.6249271 | 344.5573745 | 115.4715828 |
| $\mathbf{1 3}$ | 241.9336109 | 73.34367099 | 133.270489 | 245.0942147 |
| $\mathbf{1 4}$ | 11.31311941 | 162.0624149 | 281.9836035 | 14.71684657 |
| $\mathbf{1 5}$ | 140.6926279 | 250.7811588 | 70.69671808 | 144.3394785 |
| $\mathbf{1 6}$ | 270.0671365 | 339.499625 | 219.4098326 | 273.9621104 |
| $\mathbf{1 7}$ | 39.451645 | 68.21864668 | 8.122947153 | 43.58474227 |
| $\mathbf{1 8}$ | 168.8311535 | 156.9373906 | 156.8360617 | 173.2073742 |
| Days as Months | Longitude | Anomaly | Latitude | Elongation |
| $\mathbf{3 0}$ | 35.29146645 | 31.9494858 | 36.88052996 | 5.722408101 |
| $\mathbf{6 0}$ | 70.58293291 | 63.8989716 | 73.76105992 | 11.4448162 |
| $\mathbf{9 0}$ | 105.8743994 | 95.84845741 | 110.6415899 | 17.1672243 |
| $\mathbf{1 2 0}$ | 141.1658658 | 127.7979432 | 147.5221198 | 22.8896324 |
| $\mathbf{1 5 0}$ | 176.4573323 | 159.747429 | 184.4026498 | 28.61204051 |
| $\mathbf{1 8 0}$ | 211.7487987 | 191.6969148 | 221.2831798 | 34.33444861 |
| $\mathbf{2 1 0}$ | 247.0402652 | 223.6464006 | 258.1637097 | 40.05685671 |
| $\mathbf{2 4 0}$ | 282.3317316 | 255.5958864 | 295.0442397 | 45.77926481 |
| $\mathbf{2 7 0}$ | 317.6231981 | 287.5453722 | 331.9247697 | 51.50167291 |
| $\mathbf{3 0 0}$ | 352.9146645 | 319.494858 | 8.805299621 | 57.22408101 |
| $\mathbf{3 3 0}$ | 28.206131 | 351.4443438 | 45.68582958 | 62.94648911 |
| $\mathbf{3 6 0}$ | 63.49759745 | 23.39382962 | 82.56635954 | 68.66889721 |
| $\mathbf{D a y s}$ | Longitude | Anomaly | Latitude | Elongation |
| $\mathbf{1}$ | 13.17638222 | 13.06498286 | 13.229351 | 12.19074694 |
| $\mathbf{2}$ | 26.35276443 | 26.12996572 | 26.458702 | 24.38149387 |
| $\mathbf{3}$ | 39.52914665 | 39.19494858 | 39.688053 | 36.57224081 |
| $\mathbf{4}$ | 52.70552886 | 52.25993144 | 52.91740399 | 48.76298775 |
| $\mathbf{5}$ | 65.88191108 | 65.3249143 | 66.14675499 | 60.95373468 |
| $\mathbf{6}$ | 79.05829329 | 78.38989716 | 79.37610599 | 73.14448162 |
| $\mathbf{7}$ | 91.45488002 | 92.60545699 | 85.33522856 |  |
|  | 104.5198629 | 105.834808 | 97.52597549 |  |


| $\mathbf{9}$ | 118.5874399 | 117.5848457 | 119.064159 | 109.7167224 |
| :--- | :--- | :--- | :--- | :--- |
| $\mathbf{1 0}$ | 131.7638222 | 130.6498286 | 132.29351 | 121.9074694 |
| $\mathbf{1 1}$ | 144.9402044 | 143.7148115 | 145.522861 | 134.0982163 |
| $\mathbf{1 2}$ | 158.1165866 | 156.7797943 | 158.752212 | 146.2889632 |
| $\mathbf{1 3}$ | 171.2929688 | 169.8447772 | 171.981563 | 158.4797102 |
| $\mathbf{1 4}$ | 184.469351 | 182.90976 | 185.210914 | 170.6704571 |
| $\mathbf{1 5}$ | 197.6457332 | 195.9747429 | 198.440265 | 182.8612041 |
| $\mathbf{1 6}$ | 210.8221154 | 209.0397258 | 211.669616 | 195.051951 |
| $\mathbf{1 7}$ | 223.9984977 | 222.1047086 | 224.898967 | 207.2426979 |
| $\mathbf{1 8}$ | 237.1721021 | 235.1696915 | 238.128318 | 219.4334449 |
| $\mathbf{1 9}$ | 250.3512621 | 248.2346743 | 251.357669 | 231.6241918 |
| $\mathbf{2 0}$ | 263.5276443 | 261.2996572 | 264.5870207 | 243.8149387 |
| $\mathbf{2 1}$ | 276.7040265 | 274.3646401 | 277.8163571 | 256.0056857 |
| $\mathbf{2 2}$ | 289.8804087 | 287.4296229 | 291.0457219 | 268.1964326 |
| $\mathbf{2 3}$ | 303.0567909 | 300.4946058 | 304.275073 | 180.3871795 |
| $\mathbf{2 4}$ | 316.2331732 | 313.5595886 | 317.504424 | 292.5779265 |
| $\mathbf{2 5}$ | 329.4095554 | 326.6245715 | 330.733775 | 304.7686734 |
| $\mathbf{2 6}$ | 342.5859376 | 339.6895544 | 343.963126 | 316.9594204 |
| $\mathbf{2 7}$ | 355.7623198 | 352.7545372 | 357.192477 | 329.1501673 |
| $\mathbf{2 8}$ | 8.938702024 | 5.819520082 | 10.42182796 | 341.3409142 |
| $\mathbf{2 9}$ | 22.11508424 | 18.88450294 | 23.65117896 | 353.5316612 |
| $\mathbf{3 0}$ | 35.29146645 | 31.9494858 | 36.88052996 | 5.722408101 |
| $\mathbf{H o u r s}$ | Longitude | Anomaly | Latitude | Elongation |
| $\mathbf{1}$ | 0.5490159256 | 0.5443742858 | 0.5512229583 | 0.507947789 |
| $\mathbf{2}$ | 1.098031864 | 1.088748572 | 1.102445917 | 1.015895578 |
| $\mathbf{3}$ | 1.647047777 | 1.633122858 | 1.487002208 | 1.523843367 |
| $\mathbf{4}$ | 2.196063703 | 2.177497143 | 2.204891833 | 2.031791156 |
| $\mathbf{5}$ | 2.745079628 | 2.721871429 | 2.756114791 | 2.539738945 |
| $\mathbf{6}$ | 3.294095554 | 3.266245715 | 3.30733775 | 3.047686734 |
| $\mathbf{7}$ | 3.843111479 | 3.810620001 | 3.858560708 | 3.555634523 |
| $\mathbf{8}$ | 4.941143331 | 4.899368573 | 4.961006625 | 4.571530101 |
| $\mathbf{9}$ | 5.490159256 | 5.443742858 | 5.512229583 | 5.07947789 |
| $\mathbf{1 0}$ | 5.988117144 | 6.063452541 | 5.587425679 |  |


| $\mathbf{1 2}$ | 6.588191108 | 6.53249143 | 6.614675499 | 6.095373468 |
| :--- | :--- | :--- | :--- | :--- |
| $\mathbf{1 3}$ | 7.137207033 | 7.07668979 | 7.165898458 | 6.603321257 |
| $\mathbf{1 4}$ | 7.686222959 | 7.621240002 | 7.717121416 | 7.111269046 |
| $\mathbf{1 5}$ | 8.235238884 | 8.165614288 | 8.268344374 | 7.619216835 |
| $\mathbf{1 6}$ | 8.78425481 | 8.709988573 | 8.819567332 | 8.127164624 |
| $\mathbf{1 7}$ | 9.333270736 | 9.254362859 | 9.370790291 | 8.635112413 |
| $\mathbf{1 8}$ | 9.882286661 | 9.798737145 | 9.922013249 | 9.143060203 |
| $\mathbf{1 9}$ | 10.43130259 | 10.34311143 | 10.47323621 | 9.651007992 |
| $\mathbf{2 0}$ | 10.98031851 | 10.88748572 | 11.02445917 | 10.15895578 |
| $\mathbf{2 1}$ | 11.52933444 | 11.43186 | 11.57568212 | 10.66690357 |
| $\mathbf{2 2}$ | 12.07835036 | 11.97623429 | 12.12690508 | 11.17485136 |
| $\mathbf{2 3}$ | 12.62736629 | 12.52060857 | 12.67812804 | 11.68279915 |
| $\mathbf{2 4}$ | 13.17638222 | 13.06498286 | 12.67812804 | 11.68279915 |

## APPENDIX E

## Table of the Moon's Anomaly

In this table, I give the relationship between the angle of the moon past the apogee of the epicycle (going clockwise) to the equation of anomaly as viewed from earth. The calculation for this was shown in Section 3.3.3.

Although the distance between the earth and center of the epicycle as calculated in Section 3.3.3 was in excellent agreement with Ptolemy's value, the table here deviates from Ptolemy's more than should be expected from that minor deviation alone. This appears to be due to accumulated errors from rounding on Ptolemy's part.

Since the moon travels counter clockwise on its epicycle, if the angle is found in the first column, the anomaly will be subtractive. If it is found in the second, it is additive.

| Angle from Apogee |  | Equation of Anomaly $\left({ }^{\circ}\right)$ |
| :--- | :--- | :--- |
| $\mathbf{6}$ | $\mathbf{3 5 4}$ | 0.519 |
| $\mathbf{1 2}$ | $\mathbf{3 4 8}$ | 1.031 |
| $\mathbf{1 8}$ | $\mathbf{3 4 2}$ | 1.533 |
| $\mathbf{2 4}$ | $\mathbf{3 3 6}$ | 2.017 |
| $\mathbf{3 0}$ | $\mathbf{3 3 0}$ | 2.479 |
| $\mathbf{3 6}$ | $\mathbf{3 2 4}$ | 2.914 |
| $\mathbf{4 2}$ | $\mathbf{3 1 8}$ | 3.316 |
| $\mathbf{4 8}$ | $\mathbf{3 1 2}$ | 3.682 |
| $\mathbf{5 4}$ | $\mathbf{3 0 6}$ | 4.007 |
| $\mathbf{6 0}$ | $\mathbf{3 0 0}$ | 4.288 |
| $\mathbf{6 6}$ | $\mathbf{2 9 4}$ | 4.523 |
| $\mathbf{7 2}$ | $\mathbf{2 8 8}$ | 4.708 |
| $\mathbf{7 8}$ | $\mathbf{2 8 2}$ | 4.841 |
| $\mathbf{8 4}$ | $\mathbf{2 7 6}$ | 4.922 |
| $\mathbf{9 0}$ | $\mathbf{2 7 0}$ | 4.949 |


| $\mathbf{9 3}$ | $\mathbf{2 6 7}$ | 4.942 |
| :--- | :--- | :--- |
| $\mathbf{9 6}$ | $\mathbf{2 6 4}$ | 4.922 |
| $\mathbf{9 9}$ | $\mathbf{2 6 1}$ | 4.888 |
| $\mathbf{1 0 2}$ | $\mathbf{2 5 8}$ | 4.841 |
| $\mathbf{1 0 5}$ | $\mathbf{2 5 5}$ | 4.781 |
| $\mathbf{1 0 8}$ | $\mathbf{2 5 2}$ | 4.708 |
| $\mathbf{1 1 1}$ | $\mathbf{2 4 9}$ | 4.621 |
| $\mathbf{1 1 4}$ | $\mathbf{2 4 6}$ | 4.523 |
| $\mathbf{1 1 7}$ | $\mathbf{2 4 3}$ | 4.412 |
| $\mathbf{1 2 0}$ | $\mathbf{2 4 0}$ | 4.288 |
| $\mathbf{1 2 3}$ | $\mathbf{2 3 7}$ | 4.153 |
| $\mathbf{1 2 6}$ | $\mathbf{2 3 4}$ | 4.007 |
| $\mathbf{1 2 9}$ | $\mathbf{2 3 1}$ | 3.850 |
| $\mathbf{1 3 2}$ | $\mathbf{2 2 8}$ | 3.682 |
| $\mathbf{1 3 5}$ | $\mathbf{2 2 5}$ | 3.504 |
| $\mathbf{1 3 8}$ | $\mathbf{2 2 2}$ | 3.316 |
| $\mathbf{1 4 1}$ | $\mathbf{2 1 9}$ | 3.119 |
| $\mathbf{1 4 4}$ | $\mathbf{2 1 6}$ | 2.914 |
| $\mathbf{1 4 7}$ | $\mathbf{2 1 3}$ | 2.700 |
| $\mathbf{1 5 0}$ | $\mathbf{2 1 0}$ | 2.479 |
| $\mathbf{1 5 3}$ | $\mathbf{2 0 7}$ | 2.251 |
| $\mathbf{1 5 6}$ | $\mathbf{2 0 4}$ | 2.017 |
| $\mathbf{1 5 9}$ | $\mathbf{2 0 1}$ | 1.777 |
| $\mathbf{1 6 2}$ | $\mathbf{1 9 8}$ | 1.533 |
| $\mathbf{1 6 5}$ | $\mathbf{1 9 5}$ | 1.284 |
| $\mathbf{1 6 8}$ | $\mathbf{1 9 2}$ | 1.031 |
| $\mathbf{1 7 1}$ | $\mathbf{1 8 9}$ | 0.776 |
| $\mathbf{1 7 4}$ | $\mathbf{1 8 6}$ | 0.519 |
| $\mathbf{1 7 7}$ | $\mathbf{1 8 3}$ | 0.260 |
| $\mathbf{1 8 0}$ | $\mathbf{1 8 0}$ | 0.000 |
|  |  |  |

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[^0]:    ${ }^{1}$ While a separate date could be chosen for each celestial object, in practice, it is simpler to use the same date for all.
    ${ }^{2}$ This was an unusual choice since the primary table used to calculate forward from this date only had values for up to 810 years meaning additional math would be required by 63 CE which was actually prior to Ptolemy's own life. As such, it is likely Ptolemy chose these dates based on the work of prior Greek astronomers. In his later work, the Handy Tables, he instead created an epoch date beginning in 323 BCE [5].

[^1]:    ${ }^{3}$ For example, in the ninth century, Al-Battani revsed Ptolemy's values for the length of the year and discovered that the position of the solar apogee was changing [13]. In the 12th century, Jabir ibn Aflah wrote Islah al-Magisti (Correction of the Almagest) which offered both a philosophical critique as well as a mathematical one [2].
    ${ }^{4}$ Understanding the celestial sphere and getting more familiar with this terminology is the focus of my class, $D e$ Sphaera.

[^2]:    ${ }^{5}$ I have a comprehensive class on the mathematical techniques used: Mathematics of the Almagest. A copy of the class handout which explores the following concepts in detail with worked examples and sample problems is available at https://b3r.org/files/Mathematics-of-the-Almagest-by-Chesey.pdf

[^3]:    ${ }^{1}$ I have omitted the constellations around the ecliptic as the diagram is not to scale and would lead to incorrect assumptions about which constellations things were in.

[^4]:    ${ }^{2}$ I am performing these calculations based on 2000 because the most commonly used modern astronomical epoch, the J2000 epoch, is based on this year. As such, it will make comparing the model to the true values easier as the position of the vernal equinox on which the ecliptic longitude is based, drifts over the years.
    ${ }^{3}$ Ptomely's times are generally given no more accurately than the nearest half hour. The actual time of the solstice would have been 1:35am CST, so to replicate Ptolemy's level of accuracy, I have rounded down.
    ${ }^{4}$ This date was during daylight savings time so one hour has been subtracted and then rounded to the nearest half hour.
    ${ }^{5}$ Converted from daylight savings and rounded slightly.

[^5]:    ${ }^{6}$ This was not during daylight savings time so no adjustment needed, but slight rounding.

[^6]:    ${ }^{7}$ Converted from GMT to CST.

[^7]:    ${ }^{8}$ This is notably different than the value of $2.5^{p}$ during Ptolemy's time.

[^8]:    ${ }^{9}$ This varies from Ptolemy's result of $1.15^{\circ}$ which is to be expected since the distance to the center of the eccentre is also changed. In particular, the center of the eccentre is moved closer to earth which would diminish the effect.
    ${ }^{10} \mathrm{We}$ only used the date of the apogee in the context of checking the accuracy of our results.

[^9]:    ${ }^{11}$ Here, I am following Ptolemy's method albeit with a slightly modified diagram. However, I believe it would have been easier to determine $\angle P Z K$ from $\operatorname{arc} P K$ in Figure 2.1 and add that to $\angle H Z P$ which is equal to $\angle Z E X$ which we found. The sum of these angles would then be $\angle H Z K$ which is the angular distance of the sun from apogee on the eccentre for the autumnal equinox which is what we're ultimately after.

[^10]:    ${ }^{12}$ I have selected this date because it is the date of an eclipse we will be using to calibrate the lunar model in Chapter 3.

[^11]:    ${ }^{13}$ It should be unsurprising that this value is actually in better agreement with the true values than the one we derived from epoch. In truth, this model has been calibrated for the year 2000 since that is the year of the solstices and equinoxes we used. In this sample problem, 2003 is much closer to 2000 than 1996. Thus, the compounded errors in mean motion over those intervals are smaller.

[^12]:    ${ }^{1}$ Toomer [10] notes that this results in an error of up to $\approx 0.1^{\circ}$
    ${ }^{2}$ Indeed, the sun's motion defines the ecliptic.

[^13]:    ${ }^{3}$ I have rounded the times for mid-eclipse to the nearest quarter hour to align with the precision of observations available to Ptolemy. In addition, times were adjusted to remove daylight savings time for eclipses 1 and 3.
    ${ }^{4}$ NASA's Eclipse Explorer is somewhat misleading on this eclipse listing it as April 14. This is likely due to the eclipse starting on April 14th, but the time of mid eclipse was April 15th as can be confirmed by other sources [14].
    ${ }^{5}$ This calculation was done as an example in Section 2.5. See discussion there for analysis of accuracy.
    ${ }^{6}$ Value from Stellarium [7] of $331.87^{\circ}$.
    ${ }^{7}$ Value from Stellarium [7] of $25.07^{\circ}$.

[^14]:    ${ }^{8}$ For further discussion and a proof, refer to https://jonvoisey.net/blog/2020/08/almagest-book-iv-babylonian-eclipse-triple-geometry-radius-of-the-epicycle/

[^15]:    ${ }^{9}$ Since this will be our final context for this exercise, I have discontinued using the subscript. Besides, writing out "epicycle" as a subscript just looks dumb.

[^16]:    ${ }^{10}$ This date has been selected because we will be making use of it in the next chapter.
    ${ }^{11}$ Note that we must add an extra hour here because May 1 was during daylight savings time, so we must undo the "spring forward" to revert it to CST thereby adding the extra hour

[^17]:    ${ }^{12}$ Here the negative is indicating the model is predicting a lower ecliptic longitude than its true position.

